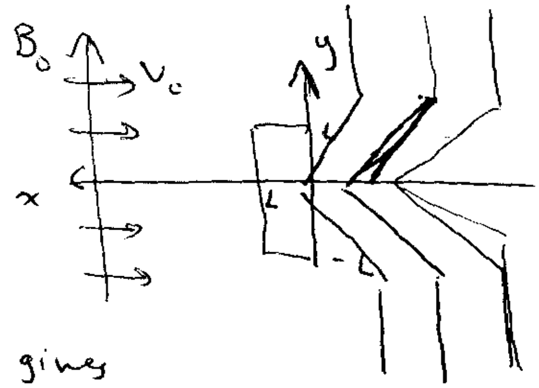
4.8.1 $\dfrac{dV}{dx} = +\left(1 - \left|\dfrac{y}{L}\right|\right) 0.9 \dfrac{V_0}{L}$



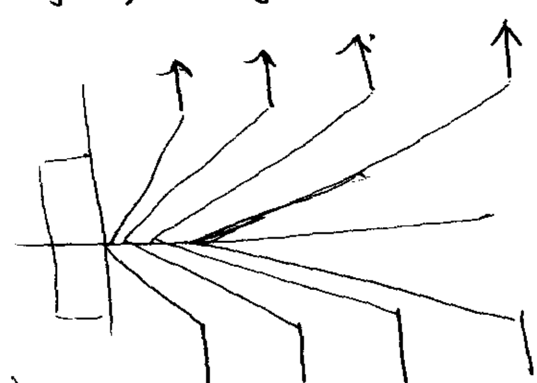
This is a linear gradient. Integrating with respect to x at constant y (treat $y>0$; $y<0$ same by symmetry) gives

$$V(x) + V_0 = +\left(1 - \left|\frac{y}{L}\right|\right) 0.9 \frac{V_0}{L} (x - L) \qquad \left[\int_{x=L}^{x} + V(x=L) = -V_0 \right]$$

So at $x=0$ (and assumed const for $x<0$) we have

$$V = -V_0 \left[1 - \left(1 - \left|\frac{y}{L}\right|\right) 0.9 \right]$$

eg $V(y=0) = -0.1 V_0$
$V(y=L) = -V_0$



a) Distance between field lines at $y > L$ is $10\times$ that at $y=0$ as field frozen to flow

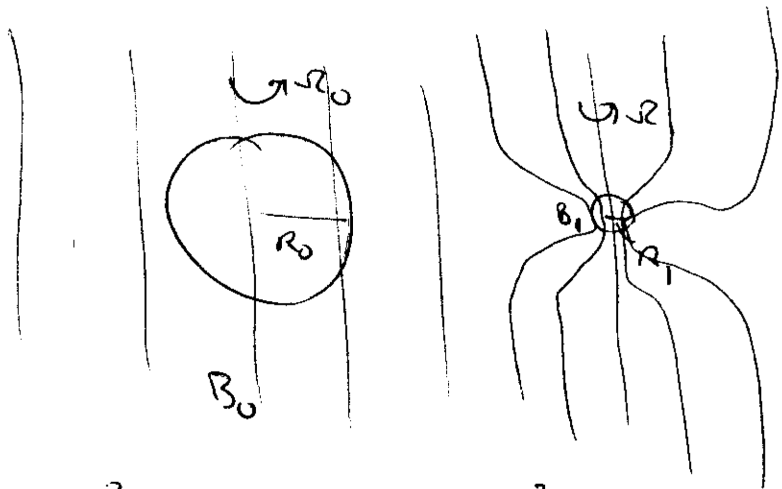
b) Per unit z-distance, in one second the same flux must pass $(0,0)$ and $(0,L)$. This flux is $\Phi_B = (|V|t)B \Rightarrow B \propto \frac{1}{|V|}$

Thus $\underline{\underline{B(0,0) = 10 B_0}}$

c) As flow decelerated, must lose energy; transferred to obstacle + also stored in magnetic energy ($B^2/2\mu_0$ has increased).

d) Field line <u>tension</u> force would <u>accelerate</u> flow in $-L<y<L$ (and also decelerate flow near $|y| \gtrsim L$).

4.8.2



a) Mass cons.

$$M = \frac{4}{3}\pi \rho_0 R_0^3 = \frac{4}{3}\pi \rho_1 R_1^3$$

$$\Rightarrow \rho_1 = \rho_0 \left(\frac{R_0}{R_1}\right)^3$$

Ang. momentum $\frac{2}{5} M R_0^2 \Omega_0 = \frac{2}{5} M R_1^2 \Omega_1 \Rightarrow \underline{\Omega_1 = \Omega_0 \left(\frac{R_0}{R_1}\right)^2}$

Thus at equatorial surface, centrifugal force/vol is

$$\rho_1 \Omega_1^2 R_1 = \rho_0 \left(\frac{R_0}{R_1}\right)^3 \Omega_0^2 \left(\frac{R_0}{R_1}\right)^4 R_0 \left(\frac{R_1}{R_0}\right) = \underline{\underline{\rho_0 \Omega_0^2 R_0 \left(\frac{R_0}{R_1}\right)^6}}$$

b) Flux freezing ⇒ flux through equatorial cut is constant, ie

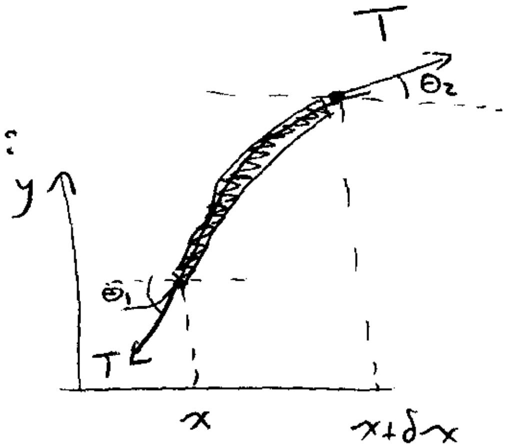
$$B_0 \pi R_0^2 = B_1 \pi R_1^2 \Rightarrow \underline{\underline{B_1 = B_0 \left(\frac{R_0}{R_1}\right)^2}}$$

Magnetic force/vol $= \left| -\nabla \frac{B^2}{2\mu_0} \right| \sim \frac{B^2}{2\mu_0} \frac{1}{R}$

$$\frac{\text{Final}}{\text{Initial}} = \frac{B_1^2/R_1}{B_0^2/R_0} = \left(\frac{B_1}{B_0}\right)^2 \frac{R_0}{R_1} = \left(\frac{R_0}{R_1}\right)^4 \frac{R_0}{R_1} = \underline{\underline{\left(\frac{R_0}{R_1}\right)^5}}$$

c) Can see comparing results in (b) & (a) that centrifugal forces increase faster (& ∴ are more likely to inhibit further collapse) than magnetic forces.

4.8.3 Deriving this from 1st principles:



mass of element of string is $\rho\,\delta x$

Newton's law for y-displacement

$$\rho\,\delta x \frac{\partial^2 y}{\partial t^2} = -T\sin\theta_1 + T\sin\theta_2$$

$$\approx -T\left(\frac{\partial y}{\partial x}\right)_x + T\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$$ for small θ (ie $\sin\theta \approx \tan\theta$)

$$= T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x$$

Thus $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{T}{\rho}\dfrac{\partial^2 y}{\partial x^2}$ which is a wave eqn $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$

with $c = \sqrt{T/\rho}$.

In the magnetic case, $T = B^2/\mu_0$ whence

$$c = \sqrt{\frac{B^2}{\mu_0\rho}} = v_A$$

4.8.4 (4.49) is

$$-\rho_0 \omega \delta \underset{\sim}{V} = -\rho_0 \frac{c_s^2}{\omega} (\underset{\sim}{k} \cdot \delta \underset{\sim}{V}) \underset{\sim}{k} - \frac{1}{\mu_0} (\underset{\sim}{k} \cdot \underset{\sim}{B_0}) \left[\frac{\underset{\sim}{k} \cdot \underset{\sim}{B_0}}{\omega} \delta \underset{\sim}{V} - \frac{\underset{\sim}{k} \cdot \delta \underset{\sim}{V}}{\omega} \underset{\sim}{B_0} \right]$$

$$+ \frac{1}{\mu_0} \left[\frac{\underset{\sim}{k} \cdot \underset{\sim}{B_0}}{\omega} (\underset{\sim}{B_0} \cdot \delta \underset{\sim}{V}) \underset{\sim}{k} - \frac{\underset{\sim}{k} \cdot \delta \underset{\sim}{V}}{\omega} B_0^2 \underset{\sim}{k} \right]$$

Take first component out of plane defined by $\underset{\sim}{k}$ + $\underset{\sim}{B_0}$. Let's call this $\hat{y}$ and let $\underset{\sim}{k} \cdot \underset{\sim}{B_0} \equiv k_{\parallel} B_0$

$$-\rho_0 \omega \delta V_y = 0 - \frac{1}{\mu_0} k_{\parallel} B_0 \frac{k_{\parallel} B_0}{\omega} \delta V_y - 0 + [0 - 0]$$

so either $\delta V_y = 0$ or $\rho_0 \omega = \frac{1}{\mu_0 \omega} k_{\parallel}^2 B_0^2$. Mult. by ω / ρ_0 to give

$$\underline{\omega^2 = k_{\parallel}^2 \frac{B_0^2}{\mu_0 \rho_0} = k_{\parallel}^2 v_A^2 = k^2 \cos^2 \Theta \, v_A^2}$$

which is 4.58 for an Alfvén Wave

For the Magnetosonic modes, follow hint by dotting 4.49 with $\underset{\sim}{k}$ + $\underset{\sim}{B_0}$

Define $\delta \underset{\sim}{V} \cdot \underset{\sim}{B_0} = \Psi_{\parallel}$; $\delta \underset{\sim}{V} \cdot \underset{\sim}{k} = \Psi_k$.

$\underset{\sim}{B_0} \cdot (4.49)$:

$$-\rho_0 \omega \Psi_{\parallel} = -\rho_0 \frac{c_s^2}{\omega} \Psi_k k_{\parallel} B_0 - \frac{1}{\mu_0} k_{\parallel} B_0 \left[\frac{k_{\parallel} B_0}{\omega} \Psi_{\parallel} - \frac{\Psi_k}{\omega} B_0^2 \right]$$

$$+ \frac{1}{\mu_0} \left[\frac{k_{\parallel} B_0}{\omega} \Psi_{\parallel} k_{\parallel} B_0 - \frac{\Psi_k}{\omega} B_0^2 k_{\parallel} B_0 \right]$$

ie

$$-\rho_0 \omega \Psi_{\parallel} = -\rho_0 \frac{c_s^2}{\omega} \Psi_k k_{\parallel} B_0 + \frac{k_{\parallel} B_0}{\mu_0} \left[-\cancel{\frac{k_{\parallel} B_0}{\omega} \Psi_{\parallel}} + \cancel{\Psi_{\parallel} \frac{k_{\parallel} B_0}{\omega}} + \cancel{\frac{\Psi_k}{\omega} B_0^2} - \cancel{\frac{\Psi_k}{\omega} B_0^2} \right]$$

or $\omega^2 \Psi_{\parallel} = c_s^2 k_{\parallel} B_0 \Psi_k$ ——————— ①

$\underset{\sim}{k} \cdot (4.49)$:

$$-\rho_0 \omega \Psi_k = -\rho_0 \frac{c_s^2}{\omega} \Psi_k k^2 - \frac{1}{\mu_0} k_{\parallel} B_0 \left[\cancel{\frac{k_{\parallel} B_0}{\omega} \Psi_k} - \cancel{\frac{\Psi_k}{\omega} k_{\parallel} B_0} \right]$$

$$+ \frac{1}{\mu_0} \left[\frac{k_{\parallel} B_0}{\omega} \Psi_{\parallel} k^2 - \frac{\Psi_k}{\omega} B_0^2 k^2 \right]$$

Use ① to eliminate $\Psi_{\parallel}$ and tidy up: (mult. by $-\frac{\omega}{\rho_0}$)

$$\omega^2 \psi_k = c_s^2 \psi_k k^2 - \frac{1}{\mu_0 \rho_0} k_\parallel B_0 \left(\frac{c_s^2 k_\parallel B_0 \psi_k}{\omega^2} \right) k^2 + \frac{1}{\mu_0 \rho_0} B_0^2 k^2 \psi_k$$

Divide out ψ_k, mult. by ω^2, and let $B_0^2 / \mu_0 \rho_0 \equiv v_A^2$

$$\omega^4 = \omega^2 k^2 c_s^2 - k_\parallel^2 k^2 c_s^2 v_A^2 + k^2 v_A^2 \omega^2$$

Collecting terms:

$$\omega^4 - \omega^2 k^2 (c_s^2 + v_A^2) + k_\parallel^2 k^2 c_s^2 v_A^2 = 0$$

Solve using quadratic formula for ω^2:

$$\omega^2 = \frac{1}{2} \left\{ k^2 (c_s^2 + v_A^2) \pm \sqrt{k^4 (c_s^2 + v_A^2)^2 - 4 k_\parallel^2 k^2 c_s^2 v_A^2} \right\}$$

$$= \frac{1}{2} \left\{ k^2 (c_s^2 + v_A^2) \pm k^2 (c_s^2 + v_A^2) \sqrt{1 - \frac{4 k_\parallel^2 k^2 c_s^2 v_A^2}{k^4 (c_s^2 + v_A^2)^2}} \right\}$$

$$= k^2 \frac{c_s^2 + v_A^2}{2} \left[1 \pm \sqrt{1 - \frac{4 c_s^2 v_A^2 \cos^2 \Theta}{(c_s^2 + v_A^2)^2}} \right]$$

which is (4.59) using $k_\parallel = k \cos \Theta$

(4.8.5) (4.59) is $\omega^2 = k^2 \frac{c_s^2 + v_A^2}{2}\left[1 \pm \sqrt{1 - \frac{4 c_s^2 v_A^2 \cos^2\theta}{(c_s^2+v_A^2)^2}}\right]$

$$\sqrt{1 - \frac{4\, {c_s^2}/{v_A^2}\cos^2\theta}{\left({c_s^2}/{v_A^2}+1\right)^2}} \approx 1 - \frac{2 c_s^2/v_A^2 \cos^2\theta}{1 + 2c_s^2/v_A^2 + c_s^4/v_A^4} \approx 1 - 2\frac{c_s^2}{v_A^2}\left(1 + O\frac{c_s^2}{v_A^2}\right)\cos^2\theta$$

$$= 1 - \frac{2c_s^2\cos^2\theta}{v_A^2} + O\, c_s^4/v_A^4$$

whence $\omega^2 = \frac{k^2 v_A^2}{2}\left(1 + \frac{c_s^2}{v_A^2}\right)\left[1 \pm \left(1 - \frac{2c_s^2}{v_A^2}\cos^2\theta\right)\right]$

Slow mode uses "$-$" root, ie $\omega^2 = \frac{k^2 \cancel{v_A^2}}{\cancel{2}}\left(1 + \frac{c_s^2}{v_A^2}\right)\cancel{2}\frac{c_s^2}{\cancel{v_A^2}}\cos^2\theta \approx k^2 c_s^2 \cos^2\theta$

so $\boxed{\omega^2 = k_z^2 c_s^2}$ $\omega = \pm k_z c_s$ and the group velocity

a) is $\frac{\partial \omega}{\partial \underset{\sim}{k}} = \left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_y}, \frac{\partial\omega}{\partial k_z}\right) = (0, 0, c_s)$

Thus the group velocity is guided along $\underset{\sim}{B_0} \equiv B_0\hat{z}$

b) From 4.48 $\delta p = \frac{\rho_0 c_s^2}{\omega}\underset{\sim}{k}\cdot\underset{\sim}{\delta V}$ $\Rightarrow \underset{\sim}{k}\cdot\underset{\sim}{\delta V} = \frac{\omega\,\delta p}{\rho_0 c_s^2}$ (*)

while from 4.46 $\underset{\sim}{B_0}\cdot\underset{\sim}{\delta B} = -\frac{\underset{\sim}{k}\cdot\underset{\sim}{B_0}}{\omega}\underset{\sim}{B_0}\cdot\underset{\sim}{\delta V} + \frac{\underset{\sim}{k}\cdot\underset{\sim}{\delta V}}{\omega}B_0^2$

$\hat{k}\cdot(4.49)$ gives $-\rho_0\omega\,\underset{\sim}{k}\cdot\underset{\sim}{\delta V} = -\rho_0\frac{c_s^2}{\omega}k^2(\underset{\sim}{k}\cdot\underset{\sim}{\delta V}) + \frac{1}{\mu_0}\left[\frac{\underset{\sim}{k}\cdot\underset{\sim}{B_0}}{\omega}\underset{\sim}{B_0}\cdot\underset{\sim}{\delta V}\,k^2 - \frac{\underset{\sim}{k}\cdot\underset{\sim}{\delta V}}{\omega}B_0^2 k^2\right]$

(See prev. question)

The term inside [ ] is $-\frac{k^2}{\mu_0}\underset{\sim}{B_0}\cdot\underset{\sim}{\delta B}$. Recollecting terms then shows

$+\rho_0\,\underset{\sim}{k}\cdot\underset{\sim}{\delta V}\left(\omega - \frac{c_s^2}{\omega}k^2\right) = +\frac{k^2}{\mu_0}\underset{\sim}{B_0}\cdot\underset{\sim}{\delta B}$. Now use (*) to reveal

$\frac{\delta p}{c_s^2}\left(\omega^2 - k^2 c_s^2\right) = k^2\,\delta|\underset{\sim}{B}|^2/\mu_0 2$ since $\delta|\underset{\sim}{B}|^2 = 2\underset{\sim}{B_0}\cdot\underset{\sim}{\delta B}$ (see question)

For slow waves with $c_s^2 \ll v_A^2$ we have $\omega^2 = k_z^2 c_s^2 = k^2 c_s^2\cos^2\theta$

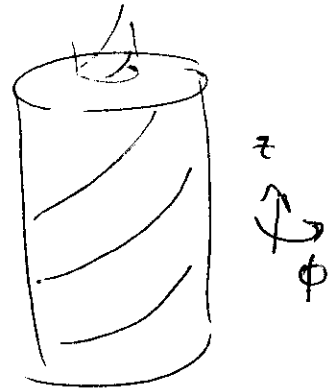
$\Rightarrow \frac{\delta p}{c_s^2}k^2 c_s^2(\cos^2\theta - 1) = -k^2\sin^2\theta\,\delta p = k^2\,\delta\left(\frac{|B|^2}{2\mu_0}\right)$ or $-\sin^2\theta\,\delta p = \delta\left(\frac{|B|^2}{2\mu_0}\right)$

$\therefore$ $\delta p \propto -\delta$(magnetic pressure) ; ie out of phase

[Fast mode in this limit has $\omega^2 = k^2 v_A^2 \gg k^2 c_s^2$ so for them

δp and δ(magnetic pressure) are in phase]

4.8.6 Given $p(r) = p_0\left(1 - \frac{r}{r_0}\right)$; B_z const



Equilib. requires (4.67)

$$\frac{d}{dr}\left[p + \frac{B_\phi^2 + B_z^2}{2\mu_0}\right] + \frac{B_\phi^2}{\mu_0 r} = 0$$

Let $P_B \equiv \frac{B_\phi^2}{2\mu_0}$ and note $\frac{dB_z^2}{dr} \equiv 0$. Using given form for $p(r)$ this gives

$$-\frac{p_0}{r_0} + \frac{dP_B}{dr} + \frac{2}{r}P_B = 0 \Rightarrow \frac{dP_B}{dr} + \frac{2}{r}P_B = +\frac{p_0}{r_0}$$

Multiply through by integrating factor $e^{\int \frac{2}{r}dr} = e^{2\ln r} = r^2$ to give

$$r^2\frac{dP_B}{dr} + 2rP_B \equiv \frac{d}{dr}(r^2 P_B) = \frac{p_0}{r_0}r^2$$

Integrating gives $r^2 P_B = \frac{p_0}{r_0}\frac{r^3}{3} + C$ so $P_B = \frac{1}{3}\frac{r}{r_0}p_0 + \frac{C}{r^2}$

Now energy density $\frac{B^2}{2\mu_0}$ must be finite [or at least $\int 2\pi r\,dr \frac{B^2}{2\mu_0}$]

which requires $C = 0$ whence $\boxed{\frac{B_\phi^2}{2\mu_0} = \frac{r}{3r_0}p_0}$ Note B gets more twisted as r increases.

Also $\underset{\sim}{J} = \frac{1}{\mu_0}\nabla \times \underset{\sim}{B} = 0\hat{r} + 0\hat{\phi} + \frac{1}{\mu_0 r}\frac{\partial}{\partial r}(rB_\phi)\,\hat{z}$ as only $B_\phi(r)$ [see A.20]

Now $B_\phi = \left(\frac{2\mu_0}{3r_0}p_0\right)^{1/2} r^{1/2}$ so $\underset{\sim}{J} = \frac{1}{\mu_0 r}\frac{\partial}{\partial r}\left[\left(\frac{2\mu_0}{3r_0}p_0\right)^{1/2} r^{3/2}\right]\hat{z}$

ie $\underset{\sim}{J} = \frac{1}{\mu_0 r}\left(\frac{2\mu_0}{3r_0}p_0\right)^{1/2}\frac{3}{2}r^{1/2}\,\hat{z}$

$$= \left(\frac{3p_0}{2\mu_0}\frac{1}{r_0 r}\right)^{1/2}\hat{z}$$