5.5.1 Solar wind eqn $\left(V^2 - \frac{2k_bT}{m}\right)\frac{1}{V}\frac{dV}{dr} = \frac{4k_bT}{mr} - \frac{GM_\odot}{r^2}$

a)

Take given solution $\frac{V^2}{2} - \frac{2k_bT}{m}\ln V = \frac{4k_bT}{m}\ln r + \frac{GM_\odot}{r} + K$

and differentiate wrt r:

$$\frac{\cancel{2}V}{\cancel{2}}\frac{dV}{dr} - \frac{2k_bT}{m}\frac{1}{V}\frac{dV}{dr} = \frac{4k_bT}{m}\frac{1}{r} - \frac{GM_\odot}{r^2}$$

ie $\left(V^2 - \frac{2k_bT}{m}\right)\frac{1}{V}\frac{dV}{dr} = \frac{4k_bT}{m}\frac{1}{r} - \frac{GM_\odot}{r^2}$ which is the solar wind eqn.

b) At $r = r_c$ $V^2 = \frac{2k_bT}{m} \equiv c_s^2$ so soln is

$$\boxed{\frac{c_s^2}{2} - c_s^2\ln c_s - 2c_s^2\ln r_c - \frac{GM_\odot}{r_c} = K}$$

This could be put back into solution to give.

$$\frac{V^2 - c_s^2}{2} - c_s^2\ln\left(\frac{V}{c_s}\right) = 2c_s^2\ln\left(\frac{r}{r_c}\right) + GM_\odot\left(\frac{1}{r} - \frac{1}{r_c}\right)$$

c) Density $n = \frac{1}{4\pi m}\frac{1}{Vr^2}$ so $\frac{n}{n_\oplus} = \left(\frac{1Au}{r}\right)^2$ since $V = $ const

$$B_r = B_{r\oplus}\left(\frac{1Au}{r}\right)^2 ;\ B_\phi = -\frac{\Omega}{V}B_{r\oplus}(1Au)^2\frac{1}{r} = B_{\phi\oplus}\left(\frac{1Au}{r}\right)$$

Planet	r(AU)	n(cm^-3)	Br(nT)	Bphi(nT)	B(nT)	psi_spiral(deg)
Earth	1.0000	6.6000	4.9497	-4.9497	7.0000	45.0000
Mercury	0.3900	43.3925	32.5427	-12.6917	34.9300	21.3058
Mars	1.5000	2.9333	2.1999	-3.2998	3.9659	56.3099
Jupiter	5.2000	0.2441	0.1831	-0.9519	0.9693	79.1145
Neptune	30.0000	0.0073	0.0055	-0.1650	0.1651	88.0908

using $B = \sqrt{B_r^2 + B_\phi^2}$ and $\tan\psi_{spiral} = -\frac{B_\phi}{B_r}$

5.5.2 At standoff $\frac{B^2}{2\mu_0} = \rho V^2 + \frac{B_{imf}^2}{2\mu_0}$

using $B_{subsolar} = \frac{M}{r^3}$ gives us

$$\frac{M^2}{\cancel{2}\mu_0}\frac{1}{r_s^6} = \rho V^2 + \frac{B_{imf}^2}{\cancel{2}\mu_0}$$ which can be re-arranged

$$\frac{M^2}{\mu_0\left[\cancel{2}\rho V^2 + \frac{B_{imf}^2}{\mu_0}\right]} = r_s^6 \quad \text{so} \quad \boxed{r_s^6 = \frac{M^2}{2\mu_0 \rho V^2 + B_{imf}^2}}$$

Now $\rho = n m_p = n(cm^{-3})\, 10^6 \frac{m^3}{cm^3}\, 1.7\times10^{-27} = 1.7\times10^{-21}\, n(cm^{-3})$ kg/m^3

$V^2 = (450\ km/s)^2 \approx 20\times10^{10} = 2\times10^{11}\ (m/s)^2$

$\mu_0 = 4\pi\times10^{-7}$

$B_{imf}^2(T^2) = 10^{-18}\, B_{imf}^2(nT)$

Planet	r(AU)	n(cm^-3)	Br(nT)	Bphi(nT)	B(nT)	psi_spiral(deg)
Earth	1.0000	6.6000	4.9497	-4.9497	7.0000	45.0000
Mercury	0.3900	43.3925	32.5427	-12.6917	34.9300	21.3058
Mars	1.5000	2.9333	2.1999	-3.2998	3.9659	56.3099
Jupiter	5.2000	0.2441	0.1831	-0.9519	0.9693	79.1145
Neptune	30.0000	0.0073	0.0055	-0.1650	0.1651	88.0908

Planet	rho(kg/m^3)	Bimf^2(T^2)	M(Tm^3)	rs^6(m)	rs(m)	rplanet(m)	rs/rp
Earth	+1.12E-20	+4.90E-17	+7.90E+15	+1.10E+46	+4.71E+07	+6.37E+06	7.40
Mercury	+7.38E-20	+1.22E-15	+4.60E+12	+5.52E+38	+2.86E+06	+2.49E+06	1.15
Mars	+4.99E-21	+1.57E-17	+1.00E+12	+3.96E+38	+2.71E+06	+3.40E+06	0.80
Jupiter	+4.15E-22	+9.40E-19	+1.50E+20	+1.07E+56	+2.18E+09	+7.14E+07	30.53
Neptune	+1.25E-23	+2.73E-20	+2.00E+17	+6.36E+51	+4.30E+08	+2.48E+07	17.35

Notes:

Mercury just holds off the solar wind

Mars' field is too weak to stand-off the solar wind

Jupiter has an enormous magnetosphere (3 $R_\odot$!)

5.5.3 Re-do derivation starting at (5.16)

$$I = 4\pi r^2 \rho V = \text{const}$$

$$\rho V \frac{dV}{dr} = -\frac{dp}{dr} - \frac{\rho G M_\odot}{r^2}$$ but $p\rho^{-\gamma} = C$

$$\Rightarrow \rho^{-\gamma}\frac{dp}{dr} - p\gamma\rho^{-\gamma-1}\frac{d\rho}{dr} = 0$$

$$= -\frac{\gamma p}{\rho}\frac{d\rho}{dr} - \frac{\rho G M_\odot}{r^2} \quad (*)$$

$$\Rightarrow \frac{dp}{dr} = +\frac{\gamma p}{\rho}\frac{d\rho}{dr}$$

$$\rho = \frac{I}{4\pi}\frac{1}{r^2 V} \Rightarrow \frac{d\rho}{dr} = \frac{I}{4\pi}\left[\frac{-2}{r^3 V} - \frac{1}{r^2V^2}\frac{dV}{dr}\right]$$

So momentum eqn becomes

$$\rho V \frac{dV}{dr} = +\frac{\gamma p}{\rho}\frac{I}{4\pi}\left[\frac{2}{r^3 V} + \frac{1}{r^2 V^2}\frac{dV}{dr}\right] - \frac{\rho G M_\odot}{r^2}$$

$$= \frac{\gamma p}{\cancel{\rho}}\cancel{r^2\rho V}\left[\frac{2}{r^{\cancel{3}}\cancel{V}} + \frac{1}{\cancel{r^2}V^{\cancel{2}}}\frac{dV}{dr}\right] - \rho\frac{GM_\odot}{r^2} = \gamma p\left[\frac{2}{r} + \frac{1}{V}\frac{dV}{dr}\right] - \frac{\rho G M_\odot}{r}$$

Divide by ρ and re-arrange:

$$\boxed{\left(V^2 - \frac{\gamma p}{\rho}\right)\frac{1}{V}\frac{dV}{dr} = \frac{\gamma p}{\rho}\frac{2}{r} - \frac{GM_\odot}{r^2}}$$

Adiabatic Solar wind Equation

To solve this we need to re-write $\frac{\gamma p}{\rho} = \gamma C\rho^{\gamma-1} =$ ~~...~~
Go back to (*) and divide by ρ to give

$$V\frac{dV}{dr} = -\gamma C\rho^{\gamma-1}\frac{1}{\rho}\frac{d\rho}{dr} - \frac{GM_\odot}{r^2} = -\gamma C\rho^{\gamma-2}\frac{d\rho}{dr} - \frac{GM_\odot}{r^2}$$

ie $\frac{d}{dr}\left(\frac{1}{2}V^2\right) = -\gamma C\frac{1}{\gamma-1}\frac{d}{dr}\left(\rho^{\gamma-1}\right) + \frac{d}{dr}\left(\frac{GM_\odot}{r}\right)$ which can be integrated to give

$$\frac{V^2}{2} + \frac{\gamma}{\gamma-1}C\rho^{\gamma-1} - \frac{GM_\odot}{r} = K \qquad C\rho^{\gamma-1} = p\rho^{-\gamma}\rho^{\gamma-1} = \frac{p}{\rho}$$

ie $$\boxed{\frac{V^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} - \frac{GM_\odot}{r} = K}$$