Queen Mary
University of London

MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date: 2nd May 2008

Time: 10:00

Course Organiser: Prof WJ Spence

Deputy CO: Dr O Soloviev

Instructions: Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

1. Maxwell's equations in linear media are

$$\nabla \cdot \mathrm{D} = \rho, \quad \nabla \times \mathrm{H} = \mathrm{J} + \frac{\partial \mathrm{D}}{\partial t}, \quad \nabla \cdot \mathrm{B} = 0, \quad \nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}.$$

(i) Consider a region of space V bounded by a closed surface S, and also let C be a closed contour in space with an open surface S' spanning the contour. Explaining the notation used, derive from the above equations the integral forms

$$\int_S \mathrm{D} \cdot d\mathrm{S} = \int_V \rho \, dV, \quad \oint_C \mathrm{H} \cdot d\mathrm{l} = \int_{S'} (\mathrm{J} + \frac{\partial \mathrm{D}}{\partial t}) \cdot d\mathrm{S}'$$

$$\int_S \mathrm{B} \cdot d\mathrm{S} = 0, \quad \oint_C \mathrm{E} \cdot d\mathrm{l} = -\int_{S'} \frac{\partial \mathrm{B}}{\partial t} \cdot d\mathrm{S}'.$$

[6 marks]

(ii) Consider two regions, labelled by $i = 1, 2$, containing different linear media, which meet at an infinite two-dimensional boundary, with unit normal n to the boundary. Let $\mathrm{E}_i, \mathrm{D}_i, \mathrm{B}_i, \mathrm{H}_i$ for $i = 1, 2$ label the electromagnetic fields in the two regions.

Using a suitable small, shallow cylinder, straddling the boundary between the two regions, with surface charge density σ, derive the boundary conditions

$$(\mathrm{D}_2 - \mathrm{D}_1) \cdot \mathrm{n} = \sigma, \quad (\mathrm{B}_2 - \mathrm{B}_1) \cdot \mathrm{n} = 0,$$

from two of the integral equations above.

Now considering a suitable small rectangle straddling the boundary, with current density K on the surface of the rectangle, derive the further boundary conditions

$$\mathrm{n} \times (\mathrm{H}_2 - \mathrm{H}_1) = \mathrm{K}, \quad \mathrm{n} \times (\mathrm{E}_2 - \mathrm{E}_1) = 0,$$

[8 marks]

(iii) Consider incident, refracted and reflected waves at this matter interface, with

$$\mathrm{E}_{inc} = \mathrm{E}_0 e^{-i(\omega t - \mathrm{k} \cdot \mathrm{x})}, \quad \mathrm{E}_{refr} = \mathrm{E}_0' e^{-i(\omega t - \mathrm{k}' \cdot \mathrm{x})}, \quad \mathrm{E}_{refl} = \mathrm{E}_0'' e^{-i(\omega t - \mathrm{k}'' \cdot \mathrm{x})}.$$

Assume that the matter interface is at $z = 0$, and that the incident wave has electric field parallel to the $z-x$ plane. Let the angles of incidence, refraction and reflection be $\theta, \theta', \theta''$ respectively. Show that the boundary conditions on the fields E at the interface imply that

$$-E_0 \cos\theta e^{ikx \sin\theta} + E_0'' \cos\theta'' e^{ikx \sin\theta''} = -E_0' \cos\theta' e^{ik'x \sin\theta'}$$

must be true for all x. Show that this implies that $\theta = \theta''$ (the law of reflection), and $k \sin\theta = k' \sin\theta'$ (Snell's law). [6 marks]

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2. For an oscillating electric dipole with strength p, oscillating in time as $e^{-i\omega t}$, the vector potential is given by

$$\mathbf{A}^{e.d.}(\mathbf{r},t) = -\frac{1}{4\pi\epsilon}\frac{e^{ikr}}{r}\frac{ik}{c}\,\mathbf{p}\,e^{-i\omega t},$$

where $\mathbf{r} = (x, y, z)$, $r = \sqrt{x^2+y^2+z^2}$ and $k = \omega/c$.

(i) Show that in the far zone, where $kr >> 1$, this results in the magnetic field

$$\mathbf{B}^{e.d.}(\mathbf{p}) = \frac{1}{4\pi\epsilon}\frac{k^2}{c}\,\mathbf{n}\times\mathbf{p}\,\frac{e^{ikr}}{r}\,e^{-i\omega t}$$

where $\mathbf{n} = \frac{1}{r}\mathbf{r}$. [4 marks]

(ii) For the electric field $\mathbf{E}^{e.d.}(\mathbf{p})$, use the source-free Maxwell equation $\dot{\mathbf{E}} = c^2\nabla\times\mathbf{B}$ and the fact that the time dependence of the fields is $e^{-i\omega t}$ to deduce that

$$\mathbf{E}^{e.d.}(\mathbf{p}) = \frac{ic}{k}\,\nabla\times\mathbf{B}^{e.d.}(\mathbf{p})\,e^{-i\omega t}$$

and hence that in the far zone

$$\mathbf{E}^{e.d.}(\mathbf{p}) = c\,\mathbf{B}^{e.d.}(\mathbf{p})\times\mathbf{n}\,.$$

[5 marks]

(iii) The vector potential for an oscillating *magnetic* dipole is given by

$$\mathbf{A}^{m.d.}(\mathbf{r},t) = ik\,\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,\mathbf{n}\times\mathbf{m}\,e^{-i\omega t}.$$

Show that this is proportional to the magnetic field for the electric dipole, with p replaced by m:

$$\mathbf{A}^{m.d.} = \frac{i}{kc}\,\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m}).$$

Thus prove that the electric and magnetic fields for a magnetic dipole are given by

$$\mathbf{B}^{m.d.}(\mathbf{m}) = \frac{1}{c^2}\,\mathbf{E}^{e.d.}(\mathbf{p}\to\mathbf{m}),$$

$$\mathbf{E}^{m.d.}(\mathbf{m}) = -\,\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m}).$$

[8 marks]

(iv) How are the polarisation vectors, the directions of the magnetic fields, and the directions of the radiation n oriented with respect to each other in the two cases of electric and magnetic dipole radiation in the far zone ?

[3 marks]

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3. (i) Show in the Lorentz gauge ($\partial^\mu A_\mu = 0$), with $A^\mu = (\frac{1}{c}\Phi, \mathrm{A})$ and $j^\mu = (c\rho, \mathrm{J})$, that the Maxwell equation $\partial^\mu F_{\mu\nu} = \mu_0 j_\nu$ reduces to

$$\partial^\mu \partial_\mu \mathrm{A} = \mu_0 \mathrm{J}, \qquad \partial^\mu \partial_\mu \Phi = \frac{1}{\epsilon_0}\rho.$$

[3 marks]

(ii) Integrate the equation for A above with $\int_{-\infty}^{\infty} e^{-i\omega t}$ to obtain the Fourier transformed equation

$$(\nabla^2 + k^2)\mathrm{A}(\mathrm{x}, \omega) = -\mu_0 \mathrm{J}(\mathrm{x}, \omega), \qquad (1)$$

with $k^2 = \omega^2/c^2$. [4 marks]

(iii) Suppose that there exists a Green function $G_k(\mathrm{x}, \mathrm{x}')$, satisfying

$$(\nabla^2 + k^2)G_k(\mathrm{x}, \mathrm{x}') = -4\pi\delta^3(\mathrm{x} - \mathrm{x}'). \quad (2)$$

Show that

$$\mathrm{A}(\mathrm{x}, \omega) = \frac{\mu_0}{4\pi} \int \mathrm{G_k}(\mathrm{x}, \mathrm{x}')\mathrm{J}(\mathrm{x}', \omega)\mathrm{d}^3\mathrm{x}'$$

solves equation (1) above. [3 marks]

(iv) Give an argument why $G_k(\mathrm{x}, \mathrm{x}')$ must be purely a function of $r = |\mathrm{r}| = |\mathrm{x} - \mathrm{x}'|$. Show that in this case equation (2) becomes

$$\frac{1}{r}\frac{d^2}{dr^2}(rG_k(r)) + k^2 G_k(r) = -4\pi\delta^3(\mathrm{r})$$

and hence that when $r \neq 0$, $G_k(r)$ is given by

$$G_k(r) = \frac{1}{r}(Ae^{ikr} + Be^{-ikr}), \quad (3)$$

for some constants A, B. [5 marks]

(v) A solution of Poisson's equation $\nabla^2\phi = -\frac{1}{\epsilon_0}\rho$ is $\phi = \frac{1}{4\pi\epsilon}\int \frac{\rho(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|}d^3\mathrm{r}'$. Use this fact to show that when $r \to 0$, (3) above remains a solution of equation (2) if

$$A + B = 1.$$

[5 marks]

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4. Consider the Maxwell equations in a vacuum with sources -

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \quad \nabla \times \mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J}.$$

(i) Show that the first two of these equations may be solved by introducing the potentials A and Φ, and writing

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Show that the other two Maxwell equations then become

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho,$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) = -\mu_0\mathbf{J}.$$

[6 marks]

(ii) Show that the definitions of the potentials are unchanged if we make the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda, \quad \Phi \to \Phi - \frac{\partial \Lambda}{\partial t}$$

for any function Λ. [2 marks]

(iii) In Lorentz covariant notation, Maxwell's equations above may be written

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad \partial^\mu F_{\mu\nu} = -\mu_0 j_\nu.$$

Show that the first of these equations is solved by writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Write down the gauge transformations on A_μ and show that they leave $F_{\mu\nu}$ invariant. [4 marks]

(iv) Consider Maxwell's equations in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Show that the equation for A can be written

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J}_t,$$

where $\mathbf{J}_t$ is transverse ($\nabla \cdot \mathbf{J}_t = 0$). You may use the result that $\nabla^2\frac{1}{|\mathbf{x}'-\mathbf{x}|} = -4\pi\delta^3(\mathbf{x}'-\mathbf{x})$ and the identities $\nabla^2\mathbf{J} = \nabla\nabla \cdot \mathbf{J} - \nabla \times \nabla \times \mathbf{J}$, and $\mathbf{J}(\mathbf{x}) = \int \delta^3(\mathbf{x}'-\mathbf{x})\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$ for any vector field J.

[8 marks]

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5. The electric and magnetic fields generated by a charged particle moving with velocity $c\beta$ and acceleration $c\dot{\beta}$ are given by the Lienard-Wiechert expressions

$$\mathbf{B} = \frac{1}{c}[\mathbf{n} \times \mathbf{E}]_{\text{ret}},$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n}-\beta}{\gamma_u^2 R^2(1-\beta\cdot\mathbf{n})^3}\right]_{\text{ret}} + \frac{q}{4\pi\epsilon_0}\frac{1}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\beta)\times\dot{\beta}]}{(1-\beta\cdot\mathbf{n})^3 R}\right]_{\text{ret}},$$

where $\mathbf{n}$ is the unit vector which points from the point on the particle trajectory to the field point $\mathbf{x}$, with $\mathbf{x} = \mathbf{r}(\tau_0) = \mathbf{n}R$, and the retarded time is $t_{ret} = t - R/c$.

(i) Show that for the case when the acceleration is parallel to the velocity, the electric field far from the charge is given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}\times\dot{\beta}]}{(1-\beta\cdot\mathbf{n})^3 R}\right]_{\text{ret}}.$$

[2 marks]

(ii) Show that in this case, the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$, far from the charge is

$$\mathbf{S} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\left[\frac{\dot{\beta}\sin\theta}{(1-\beta\cdot\mathbf{n})^3 R}\right]^2_{\text{ret}}\mathbf{n},$$

where θ is the angle between $\mathbf{n}$ and the common direction of the velocity and acceleration of the particle. [4 marks]

(iii) Show that the power radiated per unit solid angle is given by

$$\frac{dP(t')}{d\Omega} = R^2\mathbf{n}\cdot\mathbf{S}\frac{dt}{dt'}.$$

and hence equals

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}.$$

[6 marks]

(iv) For non-relativistic motion, deduce from this the Larmor formula

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\dot{u}^2\sin^2\theta.$$

[2 marks]

(v) Without making the non-relativistic approximation, show that the maximum intensity of radiation is observed at the angle

$$\theta_{\text{max}} = \cos^{-1}\left[\frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)\right].$$

[6 marks]

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Formula Sheet

$$
\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \\
\nabla \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{a}, \\
\nabla \times (\psi \mathbf{a}) &= (\nabla\psi) \times \mathbf{a} + \psi(\nabla \times \mathbf{a}), \\
\nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \\
\nabla\big(\psi(r)\big) &= \mathbf{n}\psi'(r).
\end{aligned}
$$

Maxwell's equations:

$$
\begin{aligned}
\nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{E} &= -\tfrac{\partial \mathbf{B}}{\partial t}, \\
\nabla \cdot \mathbf{D} &= \rho, & \nabla \times \mathbf{H} &= \mathbf{J} + \tfrac{\partial \mathbf{D}}{\partial t}.
\end{aligned}
$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

$$\nabla \cdot \mathbf{J} + \dot{\rho} = 0.$$

For linear isotropic media:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dx^\alpha \eta_{\alpha\beta} dx^\beta.$$

$$\eta_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta = 0 \\ -1 & \text{if } \alpha = \beta = 1, 2, 3 \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \qquad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right).$$

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial^\beta \partial_\alpha A^\alpha = \mu_0 j^\beta; \qquad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$

$$\|F^{\alpha\beta}\| = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

In spherical coordinates (r, θ, ϕ), for a scalar field $G(r, \theta, \phi)$,

$$\nabla^2 G = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial G}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 G}{\partial \phi^2}.$$

A solution of Poisson's equation $\nabla^2 \phi = -\frac{1}{\epsilon_0}\rho$ is $\phi = \frac{1}{4\pi\epsilon}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$.

End of Examination Paper **Prof WJ Spence**