Queen Mary
University of London

M.Sc. EXAMINATION

ASTM003 Angular Momentum and Accretion in Astrophysics

Friday, 26th May, 2006
18:15 – 19:45

Time Allowed: 1h 30m
This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section. Marks awarded are shown next to the questions.
Calculators are NOT permitted in this examination.

SECTION A

You should attempt ALL questions. Marks awarded are shown next to each question.

1. (a) Using the law of gravitation, show that under Keplerian motion about a central star of mass M, the angular velocity of material in an accretion disc depends on the distance R from the star as $\Omega(R) \propto R^{-3/2}$. [2 marks]
Show that derivative of the angular velocity

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = -\frac{3}{2}\frac{\Omega}{R} \ .$$ [2 marks]

What is the specific angular momentum j as a function of M and R in this Keplerian case? [1 mark]

(b) A slowly rotating gas cloud is initially supported by hydrostatic equilibrium. The specific angular momentum of gas at the edge of the cloud is j.
The cloud cools and collapses under its own gravity, forming a star of mass M and a rotating circumstellar disc. The mass of the disc is negligible compared to that of the star.
If the angular momentum of the gas is conserved during the collapse, calculate the radius of the disc in terms of j and M. [3 marks]

(c) Explain the term minimum mass solar nebula. [2 marks]

(d) A relationship between surface mass density Σ and radial distance R from the central star of the form $\Sigma \propto R^{-3/2}$ is often used in modelling the density profile of protoplanetary discs. What justification is there for this particular radial dependence? [2 marks]
If $M_D(R)$ is the mass of this protoplanetary disc interior to a radius R, what is

the dependence of $M_D(R)$ on R? [2 marks]

Hence show that $\Sigma(R) = \dfrac{M_D(R)}{4\pi R^2}$. [1 marks]

[Total 15 marks for question]

2. (a) A solid core of mass m_c has formed in a protoplanetary disc by the coagulation of planetesimals of mass m. The core has sufficient mass that its own gravitational effects on the planetesimals are significant. By considering the dynamics of the planetesimals, show that the core's cross-section for capture of these bodies is

$$A_{cap} = \pi R_c^2 \left(1 + \frac{2Gm_c}{R_c v^2}\right) ,$$

where R_c is the radius of the core, v is the typical velocity of planetesimals relative to the core, and G is the constant of gravitation. [8 marks]

Hence show that the growth rate of the core is

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} = nmv\pi R_c^2 \left(1 + \frac{2Gm_c}{R_c v^2}\right) ,$$

where n is the number density of the planetesimals in the disc. [4 marks]

Assume that the half-thickness H_s of the planetesimal disc is related to the velocity of planetesimals by $v \simeq H_s\Omega$, where Ω is the orbital angular velocity about the central star. If ρ_s is the density of the planetesimal layer in the disc, show that when the gravitational focusing dominates over the geometrical cross-section,

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} \simeq \frac{2\pi G\rho_s R_c m_c}{v^2} H_s\Omega \simeq \frac{\pi G\Sigma_s m_c^{\frac{4}{3}}}{v^2} \left(\frac{3}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} \Omega ,$$

where Σ_s is the surface density of the planetesimals in the disc and ρ_{gr} is the density of material in the planetesimals. [6 marks]

Hence show that the time taken for the protoplanetary core to grow from a size $m_c(0)$ to $m_c(t_G)$ by gravitational accretion is

$$t_G \simeq \frac{3\left(m_c(0)^{-\frac{1}{3}} - m_c(t_G)^{-\frac{1}{3}}\right)}{\pi G\Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} \frac{v^2}{\Omega}$$

[5 marks]

Argue that this time can be approximated by

$$t_G \simeq \frac{3m_c(0)^{-\frac{1}{3}}}{\pi G\Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} H_s^2\,\Omega .$$

[2 marks]

[Total 25 marks for question]

3. (a) Gas is accreted through an accretion disc on to a neutron star of radius $R = 10^4$ m and mass $M = 3 \times 10^{30}$ kg. Show, by an order of magnitude calculation, that the total energy per unit mass of accreted material that can be liberated when gas is transported from a large distance to the surface of the neutron star exceeds the equivalent figure, $\sim 10^{15}$ J kg^{-1}, that can be released by the complete nuclear fusion of hydrogen to iron. ($G \simeq 7 \times 10^{-11}$ m^3 kg^{-1}s^{-2}.) [4 marks]

The accreted material reaches the inner edge of the accretion disc very close to the surface of the neutron star, where it moves on a circular orbit of radius R_{in}. Show that if the gas falls on to a non-rotating neutron star, the kinetic energy released would be equal to the potential energy released by viscous heating of the accretion disc while moving to the inner edge of the disc from a large distance. [3 marks]

(b) The rate of viscous dissipation of energy per unit area of a Keplerian accretion disc is $\epsilon_D = \frac{9}{4}\,\Omega^2\nu\Sigma$, where $\Omega(R)$ is the angular velocity at a radius R of the disc about the central star, ν is the kinematic viscosity, and $\Sigma(R)$ is the surface density of the disc at radius R.

Show that a steady state disc will have an effective temperature at a radial distance R from the central star

$$T_{eff} = \left(\frac{9}{8}\,\frac{GM\nu\Sigma}{R^3\sigma} \right)^{\frac{1}{4}} ,$$

where M is the mass of the central star, G is the constant of gravitation, and σ is the Stefan-Boltzmann constant. [3 marks]

[Total 10 marks for question]

SECTION B

Each question carries 50 marks. You may attempt all questions but only marks for the best question will be counted.

1. (a) Consider an axisymmetric accretion disc with surface density Σ, kinematic viscosity ν, and in which forces due to pressure and self-gravity may be neglected. Derive the result

$$R\,\Sigma\,v_R \;=\; \left(\frac{\partial(R^2\Omega)}{\partial R}\right)^{-1} \frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right) \;,$$

where R is the radial coordinate in the disc, Ω is the angular velocity, and v_R is the radial component of the gas velocity. In doing this, you may assume that the torque acting in the direction of speeding up the disc, due to material interior to R, is given by

$$\mathcal{T} \;=\; -\,2\pi\,R^3\,\nu\,\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R} \;,$$

and the rate of change of the specific angular momentum j with time t is

$$\frac{\mathrm{d}j}{\mathrm{d}t} \;=\; v_R\,\frac{\partial j}{\partial R}$$

for an axisymmetric disc. [19 marks]

The continuity equation is $\dfrac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{.}(\rho\mathbf{v}) \;=\; 0$, where ρ is the density and $\mathbf{v}$ the velocity of the fluid. The divergence of a vector $\mathbf{A}$ in a cylindrical coordinate system (R,ϕ,z) is

$$\boldsymbol{\nabla}\boldsymbol{.}\mathbf{A} \;=\; \frac{1}{R}\frac{\partial(RA_R)}{\partial R} + \frac{1}{R}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

where A_R, A_ϕ and A_z are the components of $\mathbf{A}$ in the R, ϕ and z directions. Hence derive the disc surface density evolution equation in the form

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial(R^2\Omega)}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\right] \;=\; 0 \;.$$

[15 marks]

Show that if the disc is in a state of Keplerian rotation, the surface density evolution equation is

$$\frac{\partial\Sigma}{\partial t} \;=\; \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(R^{1/2}\nu\Sigma\right)\right] \;.$$

[6 marks]

(b) Using a dimensional analysis, show that the evolutionary timescale τ_{ev} of an accretion disc depends on the kinematic viscosity ν and the radius R of the disc as

$$\tau_{ev} = k \frac{R^2}{\nu} ,$$

where k is a dimensionless numerical constant. You may assume that the dimensions of kinematic viscosity are [length]^2 [time]$^{-1}$.

[5 marks]

(c) In the alpha model of viscosity, the kinematic viscosity is represented as $\nu = \alpha c_s H$, where α is a dimensionless parameter, c_s is the sound speed, and H is the half-thickness of the accretion disc. Show that kinematic viscosity in a Keplerian disc at a distance R from a central star of mass M is given by

$$\nu \simeq \alpha \sqrt{GM} \left(\frac{H}{R}\right)^2 R^{1/2} .$$

[5 marks]

[Total 50 marks for question]

2. The moment of inertia of an isolated mass of fluid is defined as

$$I = \int_V \rho\, r^2 \, \mathrm{d}V \ ,$$

where $r = |\mathbf{r}|$ is the modulus of the position vector $\mathbf{r}$ of the volume element $\mathrm{d}V$, $\rho(\mathbf{r})$ is the density, and V is a volume enclosing the fluid. The position vector is measured relative to the centre of mass of the fluid.
Show that the second derivative of the moment of inertia with respect to time t is

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 4\mathcal{K} + 2 \int_V \mathbf{r}.\mathbf{f} \, \mathrm{d}V \ ,$$

where $\mathcal{K}$ is the total internal kinetic energy of the fluid and $\mathbf{f}(\mathbf{r})$ is the force per unit volume acting on the fluid. [12 marks]

The contribution to the force per unit volume from gas pressure is $\mathbf{f} = -\boldsymbol{\nabla} P$, where P is the pressure. Hence show that the contribution from gas pressure to the $2\int_V \mathbf{r}.\mathbf{f} \, \mathrm{d}V$ term is $6\int_V P \, \mathrm{d}V$, given that the pressure P on the surface S that bounds the volume V is zero.
You may find useful the vector identities

$$(\mathbf{A}.\boldsymbol{\nabla})f \equiv \boldsymbol{\nabla}.(f\mathbf{A}) - f\,\boldsymbol{\nabla}.\mathbf{A}$$

$$\int_V (\boldsymbol{\nabla}.\mathbf{A}) \, \mathrm{d}V \equiv \int_S \mathbf{A}.\mathrm{d}\mathbf{S} \qquad \text{(Gauss's theorem)}$$

and

$$\boldsymbol{\nabla}.\mathbf{r} \equiv 3$$

for any vector $\mathbf{A}$, scalar f and position vector $\mathbf{r}$. [12 marks]

Derive an expression for the contribution to $2\int_V \mathbf{r}\cdot\mathbf{f}\,\mathrm{d}V$ from gravitational forces in terms of the total internal gravitational potential energy E_g. You may assume that $\int_V \rho(\mathbf{r})\,\mathbf{r}\cdot\boldsymbol{\nabla}\Phi\,\mathrm{d}V = -E_g$, where $\Phi(\mathbf{r})$ is the gravitational potential. [4 marks]

Hence derive the full virial theorem result for a non-magnetised cloud

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; 4\mathcal{K} \,+\, 2E_g \,+\, 6\int_V P\,\mathrm{d}V$$ [3 marks]

Neglecting radiation pressure, show that the pressure contribution to $\mathrm{d}^2I/\mathrm{d}t^2$ for an isothermal gas cloud is

$$6\int_V P\,\mathrm{d}V \;=\; \frac{6\mathcal{R}}{\mu}\,T\,M \;,$$

where T is the temperature of the gas, M is the total mass of the cloud, μ is the mean molecular mass, and $\mathcal{R}$ is the gas constant. [4 marks]

What is the condition on $\mathrm{d}^2I/\mathrm{d}t^2$ for the collapse of a cloud? [3 marks]

A spherical cloud of uniform density and radius R collapses under its own gravity to form a protostar. The cloud is isothermal with a temperature T, is non-magnetised, and is initially stationary. Using the virial theorem, show that a limit on the mass M of the cloud for collapse to occur is

$$M \;>\; \frac{5\mathcal{R}T}{\mu G}\,R \;,$$

where $\mathcal{R}$ is the gas constant, μ is the mean molecular mass, and G is the constant of gravitation. You may assume that the internal potential energy of a uniform sphere of mass M and radius R is

$$E_g \;=\; -\,\frac{3}{5}\,\frac{GM^2}{R} \;.$$ [6 marks]

Use this result to estimate to within an order of magnitude the minimum mass for a cloud of radius $R = 3\times10^{15}$ m and temperature $T = 10$ K to collapse, assuming the molecular mass is $\mu = 2$. Express this figure in solar masses. Can clouds of this size and temperature collapse to form protostars if they have stellar masses? [6 marks]

(The gas constant is $\mathcal{R} = 8.3\ \mathrm{J\,mol^{-1}\,K^{-1}}$, the gravitational constant is $G = 6.6\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$, and the mass of the Sun is $M_\odot = 2.0\times10^{30}$ kg.

[Total 50 marks for question]

3. The monochromatic radiative flux, F_ν, emitted at a particular frequency, ν, by a black–body with effective temperature T_{eff} is given by

$$F_\nu \;=\; \frac{2\pi h\nu^3}{c^2}\,\frac{1}{\exp\left(h\nu/kT_{eff}\right)-1} \;,$$

where h is the Planck constant, k is the Boltzmann constant, and c is the speed of light.
An axisymmetric accretion disc has a radial effective temperature profile given by

$$T_{eff} \; = \; \beta R^{-\alpha}$$

where α and β are positive constants. The effective temperature is T_{in} at the inner edge of the disc, and T_{out} at the outer edge. Show that the luminosity, L_ν, at a particular frequency ν, is given by

$$\tfrac{1}{2}\, L_\nu \;\; = \;\; \left(\frac{2\pi}{c}\right)^2 \frac{k^{2/\alpha}}{\alpha}\, \nu^{3-2/\alpha}\, \beta^{2/\alpha}\, h^{1-2/\alpha} \int_0^\infty \frac{x^{2/\alpha\,-\,1}}{\exp\left(x\right) - 1}\,\mathrm{d}x \;\; .$$

In deriving this expression, you should only consider frequencies ν such that $kT_{in} \gg h\nu \gg kT_{out}$. [32 marks]

A steady state Keplerian accretion disc has

$$T_{eff} \;\; = \;\; \left[\,\frac{3GM\dot{m}}{8\pi\sigma R^3}\,\right]^{\frac{1}{4}} \quad ,$$

where M is the mass of the central star and $\dot{m}$ is the rate of mass transfer through the disc.
Make a sketch of $\log L_\nu$ versus $\log \nu$ for such a disc, and explain the shape of the curve. Your sketch should contain the regimes with $h\nu > kT_{in}$ and $h\nu < kT_{out}$. [13 marks]
What is the functional relationship between luminosity L_ν and mass flow rate $\dot{m}$ in this case? [5 marks]
[Total 50 marks for question]

 [End of examination paper.]