Extrasolar Planets and Astrophysical Discs

Lecture Notes

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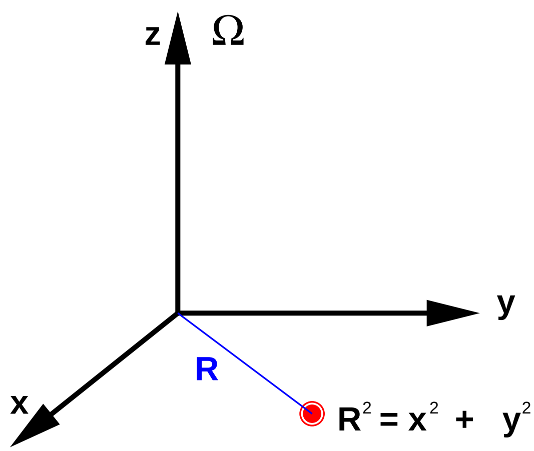
1 Lecture 1

1.1 Differentially Rotating Systems in Astrophysics

Virtually all astrophysical objects have some rotation measured in an inertial frame (*i.e.* a local frame in which Newton's Laws of motion hold in their basic forms).
For example, in the Solar System the Earth rotates with a period $P \simeq 24$ hr. The Sun rotates with a period $P \simeq 27$ days.
Solid planets rotate with uniform or solid body rotation: all points within them have the same angular velocity about the rotation axis.



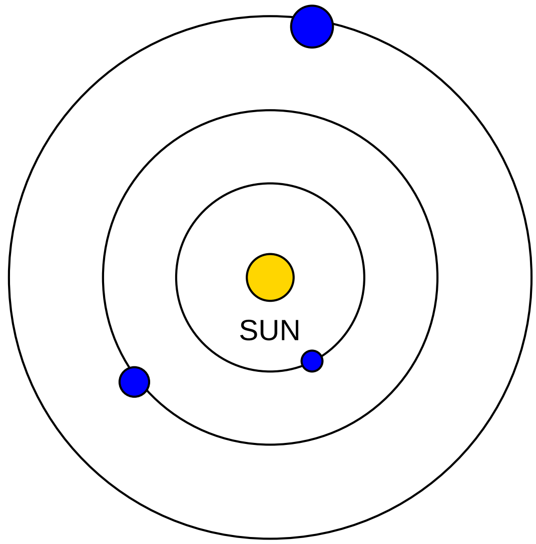
Consider a particle moving in a circle in the x–y plane of a Cartesian coordinate system (x, y, z). It rotates about the z-axis with an angular velocity Ω. The distance from the origin O is R. The components of the velocity in the x and y directions are v_x and v_y at any time. For rotation about z-axis, we have

$$v_x = -\Omega\, y \quad \text{and} \quad v_y = \Omega\, x \ ,$$

where Ω is the angular velocity. In vector notation this may be simply written as

$$\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$$

with $\mathbf{\Omega} = (0, 0, \Omega)$. $\mathbf{r}$ is the position vector of the particle measured from the origin (the origin is the centre of the circular motion), and $\mathbf{v}$ is the velocity. This vector equation is a general result for circular motion.
This particle could be one of a large number of particles in a rotating system, such as a planet, planetary system or gas disc. For such a system, we can consider Ω as a function of distance R from the centre.
For uniform rotation, $\Omega =$ constant. This is *solid-body rotation*. (Note that a gaseous body is still said to have solid-body rotation if $\Omega(R) =$ constant, even though it is not itself solid.)
For *differential rotation*, $\Omega = \Omega(R)$, (*i.e.* the angular velocity is not constant).



1.2 Systems with Differential Rotation

1.2.1 Planets in the Solar System

The system of Sun and planets can be regarded as a system in differential rotation. Planets orbit in approximately circular orbits in the same plane, with the gravity of the system dominated by the central mass (*i.e.* the Sun).
The orbital angular velocity is given by the Keplerian rotation law:

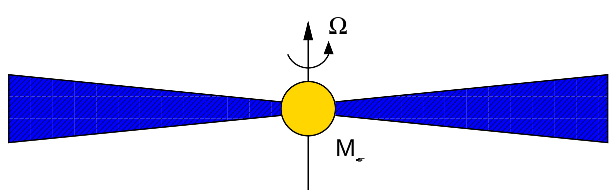
$$\Omega(R) = \sqrt{\frac{GM_\odot}{R^3}}. \tag{1}$$

Thus we see that $\Omega = \Omega(R) \propto R^{-3/2}$ such that the planets orbit the Sun in a state of differential rotation.

1.2.2 The Galaxy

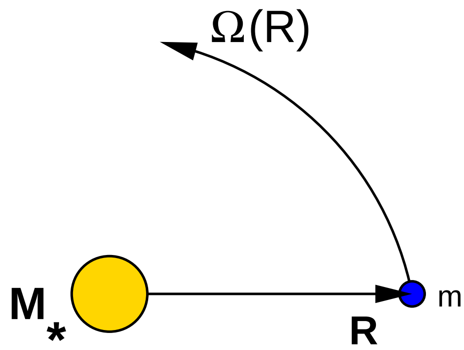
Stars orbiting in the Galactic disc resemble the Solar System, except $\Omega(R)$ differs from the Keplerian form because gravity is *not* dominated by a central mass. In this case $\Omega(R) \propto R^{-1}$ for large R, from observations, since the gravity is due to a dark matter halo that extends to large distances.

1.2.3 Accretion Discs



An accretion disc is usually a thin gaseous disc of negligible mass that orbits a central stellar mass. In this case, the disc angular velocity obeys Kepler's third law given by equation (1) to a high degree of approximation. A *Keplerian* disc is therefore one in which the central mass dominates over the mass of the disc.

Note that in each of these three main examples, the main force balance is centrifugal forces balancing gravity.



For a mass m orbiting a star of mass M_*, the gravitational acceleration towards the centre is

$$g = \frac{GM_*}{R^2} ,$$

which is balanced by the centrifugal acceleration

$$a_c = \frac{v^2}{R} = \Omega^2 R ,$$

leading to equation (1), $\Omega(R) = \sqrt{\frac{GM_*}{R^3}}$, for the angular velocity.
An important quantity used in the theory of accretion discs is the gradient $\mathrm{d}\Omega/\mathrm{d}R$ in the angular velocity. Differentiating equation (1), we get

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = -\frac{3}{2}\sqrt{\frac{GM_*}{R^5}} = -\frac{3}{2}\frac{\Omega(R)}{R} \tag{2}$$

in the Keplerian case, on substituting back for $\sqrt{GM_*}$ in terms of Ω.
Consider a body (particle) of mass m moving in a circular orbit of radius R. Its angular momentum is

$$J = m\,R^2\,\Omega \tag{3}$$

So the specific angular momentum (the angular momentum per unit mass) is

$$j = R^2\,\Omega = R^2\sqrt{\frac{GM_*}{R^3}} = \sqrt{GM_* R} \tag{4}$$

in the Keplerian case.
In general, other forces due to pressure, viscosity, etc... can be important, but are usually much smaller.
A general gaseous system (disc) obeys the fluid equations (or the Navier–Stokes equations).

1.3 Equations of Motion of an Inviscid Fluid

An *inviscid* fluid is a fluid with no viscosity. (It is also called a 'perfect fluid' or an 'ideal fluid'.) In astrophysical cases, the fluid is gas or plasma.
It is convenient to express the equations of motion in terms of the force per unit volume $\mathbf{f}$. In an inertial frame, Newton's Second Law gives,

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{f} \ , \tag{5}$$

where ρ is the density of the fluid at a point, $\mathbf{v}$ is the velocity of the fluid at that point, t is time, and $\mathbf{f}$ is the force per unit volume. However, in this context, the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\mathbf{v}\,\boldsymbol{.}\,\boldsymbol{\nabla})$$

is the convective derivative following the fluid motion (because $\mathbf{v} \equiv \mathbf{v}(\mathbf{x}, t)$). This convective derivative in a Cartesian coordinate system (x, y, z) with velocity components v_x, v_y and v_z is

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial \mathbf{v}}{\partial t} + v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \ .$$

The force per unit volume $\mathbf{f}$ in an astronomical context has contributions due to pressure, gravity, and magnetic fields (if present), and is expressed as

$$\mathbf{f} = -\boldsymbol{\nabla} P - \rho\,\boldsymbol{\nabla}\Phi + \mathbf{j} \times \mathbf{B} \ , \tag{6}$$

where P is the pressure at a point in the fluid, ρ is the density at that point, Φ is the gravitational potential, $\mathbf{j}$ is the electrical current density, and $\mathbf{B}$ is the magnetic field flux density. The first term on the right hand side of equation (6) represents the pressure force, the second term the gravitational force, and the third term the Lorentz force due to magnetic fields.
Ampère's law relates the magnetic field and the current density

$$\mathbf{j} = \frac{\boldsymbol{\nabla} \times \mathbf{B}}{\mu_0} \ , \tag{7}$$

where μ_0 is the permeability of free space. The gravitational potential at a point with position vector $\mathbf{r}$ is given by

$$\Phi(\mathbf{r}) = -\,G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V' \ , \tag{8}$$

where V is the volume that encloses the mass distribution, and $\mathrm{d}V'$ is a volume element having a position vector $\mathbf{r}'$. We assume here that the fluid is self-gravitating, that is to say that the gravitational field is caused by the fluid itself (which is a reasonable assumption for an isolated cloud).
Finally, we have the continuity equation which expresses the conservation of mass in the system,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{.}(\rho \mathbf{v}) = 0 \ . \tag{9}$$

1.4 The Virial Theorem

The Virial Theorem is a very general integral relation for a bounded fluid. In the context of star and protostellar disc formation, it is used to determine whether the combined effects of pressure, magnetic and centrifugal forces (if present) acting within a molecular cloud can stabilise it against gravitational collapse. The gravitational collapse of a rotating molecular cloud leads to the formation of a protostellar disc, such as those observed around T Tauri stars.

Consider the moment of inertia, I, of a volume of fluid defined here as

$$I = \int_V \rho r^2 \, \mathrm{d}V \ , \tag{10}$$

where $\rho(\mathbf{r}, t)$ is the mass density at a point with position vector $\mathbf{r}$ at time t, and the integral is to be performed over the volume V occupied by the fluid. (Note that this is a different definition to that used in Appendix C for the moment of inertia about a specific axis.) The position vector $\mathbf{r}$ here is measured from the centre of mass. The volume V contains the entire system, with the gas pressure $P \longrightarrow 0$ at the edge of the system.

We may re-write this integral as an integral over individual fluid elements with fixed infinitesimal mass $\mathrm{d}m$,

$$I = \int_V r^2 \, \mathrm{d}m = \int_V \mathbf{r} \cdot \mathbf{r} \, \mathrm{d}m \ . \tag{11}$$

Differentiating with respect to time we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathbf{r} \cdot \mathbf{r} \, \mathrm{d}m = \int_V \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{r} \cdot \mathbf{r}) \, \mathrm{d}m = \int_V \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \mathbf{r} + \mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right) \mathrm{d}m$$

$$\therefore \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \int_V 2\,\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \, \mathrm{d}m \ , \tag{12}$$

where $\mathrm{d}\mathbf{r}/\mathrm{d}t = \mathbf{v}$, the element velocity. Differentiating again,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V 2\,\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \, \mathrm{d}m = \int_V 2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right) \mathrm{d}m$$

$$= \int_V 2 \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{r} \cdot \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \right) \mathrm{d}m = \int_V 2 \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right|^2 \mathrm{d}m + \int_V 2\,\mathbf{r} \cdot \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \, \mathrm{d}m \ . \tag{13}$$

We note that

$$\int_V 2 \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right|^2 \mathrm{d}m = 4\mathcal{K}$$

where $\mathcal{K}$ is the total kinetic energy. The remaining term on the right-hand side of equation (13) can be expressed as the integral over volume as

$$\int_V 2\,\mathbf{r} \cdot \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \, \mathrm{d}m = \int_V 2\,\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \rho \, \mathrm{d}V = \int_V 2\,\mathbf{r} \cdot \mathbf{f} \, \mathrm{d}V \ , \tag{14}$$

using equation (5), where $\mathbf{f}$ is the force per unit volume. We now have

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 4\mathcal{K} + \int_V 2\,\mathbf{r} \cdot \mathbf{f} \, \mathrm{d}V \ . \tag{15}$$

The next step is to work out the contributions from different processes to $\mathbf{f}$.

1.4.1 Pressure Force

The pressure force per unit volume at a point in the gas is $\mathbf{f} = -\boldsymbol{\nabla}P$. Substituting into equation (15) we obtain

$$\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\,\mathrm{d}V \;=\; -\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\boldsymbol{\nabla}P\,\mathrm{d}V \;=\; -\int_V 2\,(\mathbf{r}\,\boldsymbol{.}\,\boldsymbol{\nabla})P\,\mathrm{d}V \tag{16}$$

We note that

$$(\mathbf{r}\,\boldsymbol{.}\,\boldsymbol{\nabla})P \;=\; \boldsymbol{\nabla}\,\boldsymbol{.}\,(P\mathbf{r}) \;-\; P(\boldsymbol{\nabla}\,\boldsymbol{.}\,\mathbf{r})$$

and $\boldsymbol{\nabla}\,\boldsymbol{.}\,\mathbf{r} \;=\; 3$ (see Appendix B2, including the identity for $\boldsymbol{\nabla}\,\boldsymbol{.}\,(f\mathbf{A})$ given there). Equation (16) therefore becomes

$$\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\,\mathrm{d}V \;=\; \int_V 6\,P\,\mathrm{d}V \;-\; 2\int_V \boldsymbol{\nabla}\,\boldsymbol{.}\,(P\mathbf{r})\,\mathrm{d}V \;\;. \tag{17}$$

Applying the divergence theorem (Appendix B2) to the last term in equation (17) we obtain

$$\int_V \boldsymbol{\nabla}\,\boldsymbol{.}\,(P\mathbf{r})\,\mathrm{d}V \;=\; \int_S (P\mathbf{r})\,\boldsymbol{.}\,\mathrm{d}\mathbf{S} \;=\; 0 \;, \tag{18}$$

where S is the surface of the volume V and we have used the fact that $P \to 0$ on the surface of an isolated cloud mass. Equation (17) now gives the contribution of the pressure force to the Virial Theorem to be

$$\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\,\mathrm{d}V \;=\; \int_V 6\,P\,\mathrm{d}V \;, \tag{19}$$

provided that there is zero pressure at the surface of the cloud.

1.4.2 Gravity

The acceleration due to gravity of an element of the fluid is $\mathbf{g} = -\boldsymbol{\nabla}\Phi$, where $\Phi(\mathbf{r})$ is the potential at the point with position vector $\mathbf{r}$. Therefore the gravitational force per unit volume is $\mathbf{f} = \rho\,\mathbf{g} = -\,\rho\,\boldsymbol{\nabla}\Phi$. The gravitational contribution to the virial results at any point $\mathbf{r}$ is therefore

$$\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\,\mathrm{d}V \;=\; -\int_V 2\,\rho(\mathbf{r})\,\mathbf{r}\,\boldsymbol{.}\,\boldsymbol{\nabla}\Phi\,\mathrm{d}V \;\;. \tag{20}$$

Now the gravitational potential at a position $\mathbf{r}$ inside (or outside) an isolated distribution of mass is given by

$$\Phi(\mathbf{r}) \;=\; -\,G\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \;, \tag{21}$$

and the gradient of this is

$$\boldsymbol{\nabla}\Phi(\mathbf{r}) \;=\; G\int_V \frac{\rho(\mathbf{r}')\,(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V' \;\;. \tag{22}$$

Substituting equation (22) into (20) gives

$$-\,2\int_V \rho\,\mathbf{r}\,\boldsymbol{.}\,\boldsymbol{\nabla}\Phi\;\mathrm{d}V \;=\; -\,2G\int_V\int_V \rho(\mathbf{r})\,\rho(\mathbf{r}')\,\mathbf{r}\,\boldsymbol{.}\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\;\mathrm{d}V'\,\mathrm{d}V\;. \qquad (23)$$

By splitting the right hand side of equation (23) into two equal terms, and then interchanging $\mathbf{r}$ and $\mathbf{r}'$, we can express it as

$$-\,G\int_V\int_V \rho(\mathbf{r})\,\rho(\mathbf{r}')\,\mathbf{r}\,\boldsymbol{.}\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\;\mathrm{d}V'\,\mathrm{d}V \;-\; G\int_V\int_V \rho(\mathbf{r})\,\rho(\mathbf{r}')\,\mathbf{r}\,\boldsymbol{.}\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\;\mathrm{d}V'\,\mathrm{d}V$$

$$=\;-\,G\int_V\int_V \rho(\mathbf{r})\,\rho(\mathbf{r}')\,\mathbf{r}\,\boldsymbol{.}\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\;\mathrm{d}V'\,\mathrm{d}V \;-\; G\int_V\int_V \rho(\mathbf{r}')\,\rho(\mathbf{r})\,\mathbf{r}'\,\boldsymbol{.}\,\frac{(\mathbf{r}'-\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|^3}\;\mathrm{d}V\,\mathrm{d}V'$$

$$=\;-\,G\int_V\int_V \rho(\mathbf{r})\,\rho(\mathbf{r}')\,(\mathbf{r}-\mathbf{r}')\,\boldsymbol{.}\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\;\mathrm{d}V'\,\mathrm{d}V$$

which may be re-written in the form

$$=\;-\,G\int_V\int_V \frac{\rho(\mathbf{r})\,\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;\mathrm{d}V\,\mathrm{d}V' \;=\; 2\,E_g\;, \qquad (24)$$

where E_g is the gravitational potential energy.

1.4.3 Magnetic Fields

The force per unit volume caused by magnetic fields is $\mathbf{f}=\mathbf{j}\times\mathbf{B}$, where $\mathbf{j}$ is the current density and $\mathbf{B}$ is the magnetic flux density. The magnetic contribution to the Virial Theorem is therefore

$$\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\;\mathrm{d}V \;=\; \int_V 2\,\mathbf{r}\,\boldsymbol{.}\,(\mathbf{j}\times\mathbf{B})\;\mathrm{d}V\;. \qquad (25)$$

If we use the fact that $\mathbf{r}\,\boldsymbol{.}\,(\mathbf{j}\times\mathbf{B})=\mathbf{j}\,\boldsymbol{.}\,(\mathbf{B}\times\mathbf{r})=-\,\mathbf{j}\,\boldsymbol{.}\,(\mathbf{r}\times\mathbf{B})=-\,(\mathbf{r}\times\mathbf{B})\,\boldsymbol{.}\,\mathbf{j}$, and use Ampère's law given by equation (7) to eliminate $\mathbf{j}$, then we can rewrite equation (25) as

$$\int_V 2\,\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\;\mathrm{d}V \;=\; -\,2\int_V (\mathbf{r}\times\mathbf{B})\,\boldsymbol{.}\,\mathbf{j}\;\mathrm{d}V \;=\; -\,\frac{2}{\mu_0}\int_V (\mathbf{r}\times\mathbf{B})\,\boldsymbol{.}\,(\boldsymbol{\nabla}\times\mathbf{B})\;\mathrm{d}V\;. \qquad (26)$$

Using the vector identity $\boldsymbol{\nabla}\,\boldsymbol{.}\,(\mathbf{A}\times\mathbf{B}) \;=\; -\,\mathbf{A}\,\boldsymbol{.}\,(\boldsymbol{\nabla}\times\mathbf{B}) \;+\; \mathbf{B}\,\boldsymbol{.}\,(\boldsymbol{\nabla}\times\mathbf{A})$ (see Appendix B2), and replacing $\mathbf{A}$ by $(\mathbf{r}\times\mathbf{B})$, we can rewrite equation (26) in the form

$$\int_V 2\mathbf{r}\,\boldsymbol{.}\,\mathbf{f}\;\mathrm{d}V \;=\; \frac{2}{\mu_0}\int_V \Big\{\,\boldsymbol{\nabla}\,\boldsymbol{.}\,[(\mathbf{r}\times\mathbf{B})\times\mathbf{B}] \;-\; \mathbf{B}\,\boldsymbol{.}\,[\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{B})]\,\Big\}\;\mathrm{d}V\;. \qquad (27)$$

We can express

$$\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{B}) \;=\; \mathbf{r}(\boldsymbol{\nabla}\,\boldsymbol{.}\,\mathbf{B}) \;-\; \mathbf{B}(\boldsymbol{\nabla}\,\boldsymbol{.}\,\mathbf{r}) \;+\; (\mathbf{B}\,\boldsymbol{.}\,\boldsymbol{\nabla})\mathbf{r} \;-\; (\mathbf{r}\,\boldsymbol{.}\,\boldsymbol{\nabla})\mathbf{B}$$

(see Appendix B2). Using $\boldsymbol{\nabla}\boldsymbol{.}\mathbf{B} = 0$ (one of Maxwell's Equations), $\boldsymbol{\nabla}\boldsymbol{.}\mathbf{r} = 3$ (see Appendix B2), and $(\mathbf{B}\boldsymbol{.}\boldsymbol{\nabla})\mathbf{r} = \mathbf{B}$ (Appendix B2), we obtain

$$\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{B}) \;=\; -\,2\mathbf{B} \;-\; (\mathbf{r}\boldsymbol{.}\boldsymbol{\nabla})\mathbf{B} \;\; .$$

Using this result, the second term in the integral on the r.h.s. of equation (27) can now be written as

$$\begin{aligned} -\,\mathbf{B}\boldsymbol{.}[\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{B})] \;&=\; -\,\mathbf{B}\boldsymbol{.}[-\,2\mathbf{B} \;-\; (\mathbf{r}\boldsymbol{.}\boldsymbol{\nabla})\mathbf{B}] \\ &=\; 2\mathbf{B}^2 \;+\; \mathbf{B}\boldsymbol{.}(\mathbf{r}\boldsymbol{.}\boldsymbol{\nabla})\mathbf{B} \\ &=\; 2\mathbf{B}^2 \;+\; (\mathbf{r}\boldsymbol{.}\boldsymbol{\nabla})\left(\frac{\mathbf{B}^2}{2}\right) \;\; , \end{aligned} \tag{28}$$

using the identity $(\mathbf{r}\boldsymbol{.}\boldsymbol{\nabla})\mathbf{B}^2 \;\equiv\; 2\,\mathbf{B}\boldsymbol{.}(\mathbf{r}\boldsymbol{.}\boldsymbol{\nabla})\mathbf{B}$ (see Appendix B2).
We have the vector identity $\boldsymbol{\nabla}\boldsymbol{.}(f\mathbf{A}) \equiv f\boldsymbol{\nabla}\boldsymbol{.}\mathbf{A} + (\mathbf{A}\boldsymbol{.}\boldsymbol{\nabla})f$ (Appendix B2). Replacing $\mathbf{A}$ by $\mathbf{r}$ and f by $\mathbf{B}^2/2$ and rearranging, the last term in equation (28) may be re-expressed in the form

$$\begin{aligned} (\mathbf{r}.\boldsymbol{\nabla})\left(\frac{\mathbf{B}^2}{2}\right) \;&=\; -\,\frac{\mathbf{B}^2}{2}(\boldsymbol{\nabla}\boldsymbol{.}\mathbf{r}) \;+\; \boldsymbol{\nabla}\boldsymbol{.}\left(\frac{\mathbf{B}^2}{2}\,\mathbf{r}\right) \\ &=\; -\,\frac{3\mathbf{B}^2}{2} \;+\; \boldsymbol{\nabla}\boldsymbol{.}\left(\mathbf{r}\,\frac{\mathbf{B}^2}{2}\right) \end{aligned} \tag{29}$$

using the result $\boldsymbol{\nabla}\boldsymbol{.}\mathbf{r} = 3$.
Collecting terms, we finally obtain an expression for equation (26) in the form

$$\begin{aligned} \int_V 2\,\mathbf{r}\boldsymbol{.}\mathbf{f}\,\mathrm{d}V \;&=\; \frac{2}{\mu_0}\int_V \left(\boldsymbol{\nabla}\boldsymbol{.}\,[\,(\mathbf{r}\times\mathbf{B})\times\mathbf{B}\,] \;+\; 2\,\mathbf{B}^2 \;-\; \frac{\mathbf{3B}^2}{2} \;+\; \boldsymbol{\nabla}\boldsymbol{.}\left(\mathbf{r}\,\frac{\mathbf{B}^2}{2}\right)\right)\,\mathrm{d}V \\ &=\; \frac{2}{\mu_0}\int_V \boldsymbol{\nabla}\boldsymbol{.}\left[(\mathbf{r}\times\mathbf{B})\times\mathbf{B} \;+\; \frac{\mathbf{rB}^2}{2}\right]\mathrm{d}V \;+\; \frac{1}{\mu_0}\int_V \mathbf{B}^2\,\mathrm{d}V \;\; . \end{aligned} \tag{30}$$

The last term on the r.h.s. of equation (30) is equal to twice the magnetic energy in the fluid, $\mathcal{M}$, and the first term may be re-written as a surface integral using the divergence theorem (Appendix B2), giving

$$\int_V 2\,\mathbf{r}\boldsymbol{.}\mathbf{f}\,\mathrm{d}V \;=\; \frac{2}{\mu_0}\int_S \left[(\mathbf{r}\times\mathbf{B})\times\mathbf{B} \;+\; \frac{\mathbf{rB}^2}{2}\right]\boldsymbol{.}\,\mathrm{d}\mathbf{S} \;+\; 2\mathcal{M} \;\; , \tag{31}$$

where the surface S bounds the fluid. Thus, the full Virial Theorem may be expressed as

$$\boxed{\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; 4\mathcal{K} \;+\; 2\mathcal{M} \;+\; 2E_g \;+\; 6\int_V P\,\mathrm{d}V \;+\; \mathcal{M}_s \;\; ,} \tag{32}$$

where we have denoted the surface integral on the r.h.s. of equation (31) by the symbol $\mathcal{M}_s$.

We will now apply the Virial Theorem to a uniform, isolated, isothermal, non–magnetic, spherical molecular cloud which is uniformly rotating about its z axis with angular velocity Ω. The origin is at the centre of mass. Uniform means that the density ρ is constant throughout the cloud. Isothermal means that the temperature T is the same throughout. Non-magnetic means that the magnetic flux density $B = 0$ throughout. The moment of inertia of the cloud about the axis of rotation is

$$I = \frac{2}{5} M R^2 ,$$

where M is the total mass and R is the radius, because it is a sphere of constant density (see Appendix C6). The rotational kinetic energy is therefore

$$\mathcal{K} = \frac{1}{2} I \Omega^2 = \frac{1}{5} M R^2 \Omega^2 , \tag{33}$$

on substituting for I. The internal gravitational potential energy of a uniform sphere is

$$E_g = -\frac{3}{5} \frac{GM^2}{R} , \tag{34}$$

a standard result, where G is the constant of gravitation ($G = 6.673 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$). The pressure contribution to the Virial Theorem can be determined using the ideal gas law which gives the pressure at a point as

$$P = \frac{\mathcal{R}}{\mu} \rho T , \tag{35}$$

where ρ is the density of the gas, T is the (absolute) temperature, μ is the mean molecular mass, and $\mathcal{R}$ is the gas constant ($\mathcal{R} = 8.3145$ J mol^{-1} K^{-1}). The pressure contribution to the cloud of volume V is therefore

$$6 \int_V P \, \mathrm{d}V = 6 \int_V \frac{\mathcal{R}}{\mu} \rho T \, \mathrm{d}V = 6 \frac{\mathcal{R}}{\mu} \rho T \int_V \mathrm{d}V = 6 \frac{\mathcal{R}}{\mu} \rho T V = 6 \frac{\mathcal{R}}{\mu} M T \tag{36}$$

because the density $\rho = M/V$. There is no magnetic field in the cloud, so the magnetic energy is $\mathcal{M} = 0$ and the magnetic surface integral is $\mathcal{M}_S = 0$. The Virial Theorem then becomes

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \frac{4}{5} M R^2 \Omega^2 - \frac{6}{5} \frac{GM^2}{R} + 6 \frac{\mathcal{R}}{\mu} MT \tag{37}$$

The condition for collapse of the cloud is that the moment of inertia I decreases with time and that the rate of change of I speeds up with time, so therefore,

$$\boxed{\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} < 0 .} \tag{38}$$

This enables us to solve for the mass M as

$$M > \frac{2R^3\Omega^2}{3G} + \frac{5\mathcal{R}TR}{\mu G} . \tag{39}$$

This is a limit on the mass of the cloud for collapse to occur.

As an example, consider a uniform, isolated, isothermal, non–magnetic, spherical molecular cloud which is uniformly rotating with an angular velocity $\Omega = 10^{-13}$ rad s^{-1}.

The cloud has a temperature $T = 10\,\mathrm{K}$, radius $R = 3 \times 10^{15}\,\mathrm{m} = 20\,000$ AU, and molecular mass $\mu = 2$. Putting these into equation (39), we obtain

$$M \;>\; 1.2 \times 10^{31}\ \mathrm{kg}$$

(Note that $0.002\,\mathrm{kg\ mol}^{-1}$ is used for the molecular mass μ in these calculations, because molecular masses are defined in terms of grammes, not kilogrammes, per mole.) Converting to solar masses, using $1 M_\odot = 1.989 \times 10^{30}$ kg, we find the limit is

$$M \;>\; 6.1\, M_\odot \;.$$

This is approximately stellar mass. So clouds with masses similar to stars can collapse under their own gravity. They may fragment during collapse to form more one star.