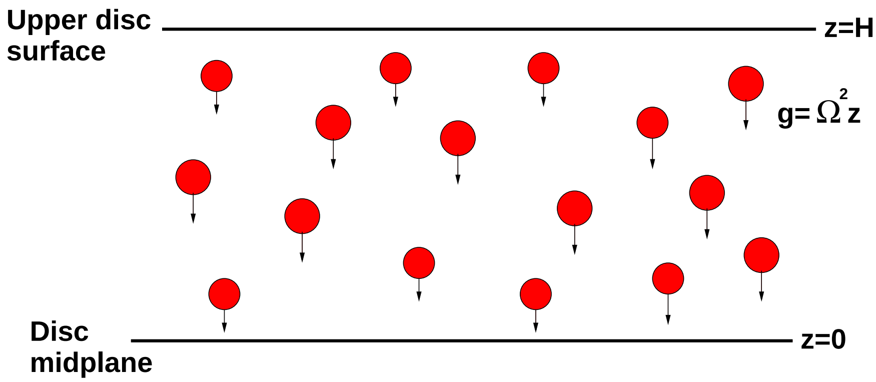
7 Week 10

7.1 The growth and settling of dust grains

The basic idea of how planetary formation occurs involves the build-up of large, rocky planetary cores/embryos by coagulation of the solid material in the protoplanetary disc. Initially, this involves the sticking together of dust grains, that gradually begin to settle towards the midplane of the disc as they grow. Dust then coagulates to form planetesimals, which in turn grow through mutual accretion into planetary embryos whose primary mode of growth is accretion from the planetesimal swarm.
We shall now consider the settling of dust grains towards the central plane of the protoplanetary disc.



We can estimate the settling time in a quiescent (non-turbulent) disc. The component of the gravitational force in the z direction acting on a dust grain of mass m_{gr} is

$$\frac{GM\, m_{gr}\, z}{r^3}$$

where M is the mass of the central star and r is the distance of the grain from the star. We may approximate $r \simeq R$ and use the Keplerian result that $GM/R^3 = \Omega^2$, where Ω is the angular velocity at the gas about the central star. The acceleration towards the midplane of the disc due to gravity is therefore

$$g_z \;=\; -\,\Omega^2 z \;\; .$$

The dust will therefore fall towards the central plane when we view the system in a rotating frame. As the dust grains move through the gas, the gas will exert a drag force on the grains, resisting their fall to the central plane. As a result, the grains will have a maximum velocity – their terminal velocity.
The drag force F_{drag} acting on a spherical particle smaller than the mean free path of gas molecules moving with a speed much smaller than the sound speed is

$$F_{drag} \;=\; \pi a^2 \rho\, c_s\, v$$

where a is the radius of the particle, ρ is the density of the gas, c_s is the sound speed of the gas and v is the velocity of the grain relative to the gas. The equation of motion for a grain of mass m_{gr} then becomes

$$m_{gr}\,\frac{\mathrm{d}v}{\mathrm{d}t} \;=\; F_{drag} \,+\, m_{gr}\, g_z \;\; .$$

If the material of the grain has an internal density ρ_{gr}, this becomes

$$\frac{4\pi a^3 \rho_{gr}}{3} \frac{\mathrm{d}v}{\mathrm{d}t} = \pi a^2 \rho \, c_s v - \frac{4\pi}{3} a^3 \rho_{gr} \, \Omega^2 z \ , \tag{144}$$

where ρ is the gas density.
In practice, a grain will quickly reach the terminal velocity, so it will fall to the central plane with the terminal velocity for most of its journey through the disc. From equation (144), using the result that $\mathrm{d}v/\mathrm{d}t = 0$ when the terminal velocity is reached, we find that the terminal velocity is

$$v = \frac{4a\rho_{gr} \, \Omega^2 H}{3\rho c_s} \sim \frac{4a\rho_{gr} \, \Omega}{3\rho} \sim \frac{8a\rho_{gr} \, \Omega H}{3\Sigma} \tag{145}$$

where we have set $z = H$, used $c_s \simeq H\Omega$ from equation (60), and have approximated $\Sigma = 2H\rho$. The settling time to the central plane is

$$\tau_s \sim H/v \sim \frac{3\Sigma}{8a\rho_{gr}\Omega} \sim \frac{\Sigma}{16a\rho_{gr}} \ \text{orbits} \ . \tag{146}$$

For $\Sigma \sim 10^2$ g cm^{-2} $= 10^3$ kg m^{-2}, $\rho_{gr} \sim 1$ g cm^{-3} $= 10^3$ kg m^{-3}, $a = 10$ μm $= 10^{-5}$ m we get $\tau_s \sim 10^4$ orbits $= 10^5$ yr at 5 AU.

As the grains settle towards the midplane of the disc, they collide with other grains, stick together, and grow. Detailed calculations show that grains grow to centimetre sizes once they reach the disc midplane, so that the solid material forms a dense dust/pebble layer within the disc. Further collisions and coagulation between the solid material within this layer causes continued growth, and current estimates suggest that large objects with sizes of $a \gtrsim 1 - 10$ km may be formed on a time scale similar to the grain settling time. Such objects are called planetesimals. Once such large objects have been formed, the effect of gravitational focusing becomes important when calculating the rate at which such objects collide to form larger objects.

7.2 Growth of planetesimals into planetary embryos

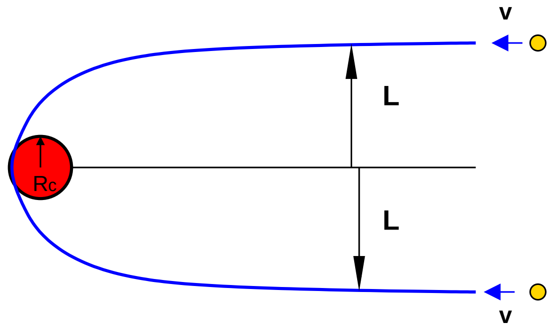
We will consider the growth of a large core from a background of smaller objects. Let the core mass be m_c, the core radius be R_c, the background object/planetesimal mass be m, and the number density of background objects be n. The accretion rate is

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} = n \, m \, v \, \pi \, R_c^2 \left(1 + \frac{2Gm_c}{R_c v^2} \right) \ , \tag{147}$$

where v is the velocity dispersion of the background objects. The term in brackets $1 + 2Gm_c/(R_c v^2)$ on the r.h.s. of equation (147) is the gravitational focusing factor.

Consider the situation drawn in the diagram overleaf. An incoming object has velocity v and impact parameter L. The condition for the object being accreted is that it just grazes the accreting core, m_c, so that its distance of closest approach must be R_c. Conservation of energy requires the velocity of the object at closest approach to be

$$\sqrt{v^2 + \frac{2Gm_c}{R_c}}$$



and conservation of angular momentum requires that

$$L\,v \;=\; R_c\sqrt{v^2+\frac{2Gm_c}{R_c}}$$

so that

$$L^2 \;=\; R_c^2\left(1+\frac{2Gm_c}{R_cv^2}\right).$$

L is now the effective radius for collisions, and leads to the expression (147) for the mass accretion rate by the accreting core.

Let us now consider two different growth regimes, the so-called ‘orderly’ growth and ‘runaway’ growth regimes.

First, let us suppose that $2Gm_c/R_c \simeq v^2$, such that the growing planetary embryo is responsible for stirring up the local planetesimal swarm (which it will do when it becomes massive compared to the background planetesimals), then we can write:

$$\frac{dm_c}{dt} = 2nm\pi vR_c^2. \tag{148}$$

Writing the embryo radius in terms of its mass and density

$$R_c^2 = \left(\frac{3m_c}{4\pi\rho_c}\right)^{2/3}$$

we obtain

$$\frac{dm_c}{dt} = 2nm\pi v\left(\frac{3}{4\pi\rho_c}\right)^{2/3} m_c^{2/3}. \tag{149}$$

We can write the mass-doubling time scale as

$$\tau_{double} = m_c\left(\frac{dm_c}{dt}\right)^{-1}$$

and we obtain $\tau_{double} \propto m_c^{1/3}$ such that lower mass embryos will double their masses on shorter time scales than larger embryos.

If we work in terms of the planetary embryo radius, R_c, then we can write

$$4\pi R_c^2\rho_c\frac{dR_c}{dt} = 2nm\pi vR_c^2 \tag{150}$$

which simplifies to

$$\frac{dR_c}{dt} = \left(\frac{\pi n m v}{2\rho_c}\right) = \text{constant}. \tag{151}$$

Thus we have that $R_c \propto t$ and the planetesimal radius grows linearly with time. This is referred to as 'orderly growth', and occurs when the velocity dispersion of the background planetesimals is high or comparable to the escape velocity from the accreting embryo.

Let us now suppose that $2Gm_c/R_c \gg v^2$, so that the gravitational focusing factor becomes important, then equation (147) may be approximated by

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} = 2nmv \, \frac{\pi R_c G m_c}{v^2} \; . \tag{152}$$

If we write the embryo radius in terms of its mass and density then we get

$$\frac{dm_c}{dt} = \frac{2\pi G n m}{v} \left(\frac{3}{4\pi\rho_c}\right)^{1/3} m_c^{4/3} \tag{153}$$

and we can see that the mass-doubling time

$$\tau_{double} = m_c \left(\frac{dm_c}{dt}\right)^{-1} \propto m_c^{-1/3}. \tag{154}$$

In this growth regime, the more massive an embryo is, the faster it doubles its mass. This mode of growth is called 'runaway growth', because planetesimals which are slightly more massive than their surrounding neighbours can grow very rapidly and detach themselves from the background mass distribution. This is how a swarm of planetesimals of initially similar masses can evolve into a system consisting of planetary embryos embedded in a planetesimal swarm.

We now consider the growth time scale in the runaway growth regime in more detail. We can write $nm = \rho_s$, the density of solids, in the disc and $v \simeq H_s\Omega$, where H_s is the thickness of the solid layer. Equation (152) may now be written as

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} = \frac{\pi G \Sigma_s m_c^{4/3}}{v^2} \left(\frac{3}{4\pi\rho_c}\right)^{1/3} \Omega \; , \tag{155}$$

where we have made the approximation $\rho_s = \Sigma_s/(2H_s)$, where Σ_s = surface density of solids, and have replaced the core radius R_c in terms of its mass, m_c and density ρ_c. Integrating equation (155) gives

$$3\left[m_c^{-1/3}(0) - m_c^{-1/3}(t)\right] = \frac{\pi G \Sigma_s \Omega}{v^2} \left(\frac{3}{4\pi\rho_c}\right)^{1/3} t \tag{156}$$

which shows that the time for the core to grow from an initial mass $m_c(0)$ to a larger mass at time t, denoted by $m_c(t)$, depends inversely on the initial core mass. The time required to grow to masses much larger than $m_c(0)$ may be approximated from equation (156) to be

$$t \simeq \frac{3m_c^{-1/3}(0)}{\pi G \Sigma_s} \left(\frac{4\pi\rho_c}{3}\right)^{1/3} H_s^2 \Omega \tag{157}$$

where we have used $v = H_s\Omega$. If we take the values $\rho_c \sim 1$ gm cm^{-3} (10^3 kg m^{-3}) and $\Sigma_s \sim 1$ gm cm^{-2} (10 kg m^{-2}), (*i.e.* gas to dust ratio = 100), then we can write

$$t \sim 10^{10} \left(\frac{m_c}{m_{Earth}}\right)^{-1/3} \left(\frac{H_s}{R}\right)^2 \left(\frac{M_*}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^{1/2} \text{ years.} \tag{158}$$

Suppose that we have 10 km sized planetesimals with $m_c \simeq 10^{18}$ g $= 10^{15}$ kg, and further assume that the dispersion velocity of the planetesimals is equal to the escape velocity from their surface (i.e. the planetesimal velocities are stirred by mutual gravitational interactions). In this case we find that $H_s/R \sim 10^{-4}$, and the growth time of a core of similar mass in orbit around a solar mass star at 1 AU is

$$t \sim 2 \times 10^5 \text{ yr.}$$

This number is significantly shorter than the expected life times of protostellar discs, as required.

7.3 The isolation mass

How large can a planetary embryo grow by accreting from the surrounding planetesimal swarm ? This question is answered by consideration of the feeding zone of an embryo, defined to be the surrounding annulus in the disc from which it is able to accrete planetesimals. The size of this annulus is set by the maximum distance over which the planetary embryo's gravity is able to perturb planetesimal orbits so that they can collide. We define the half-width of this annulus by $\Delta a_{max} = C.R_{Hill}$ where C is a constant and R_{Hill} is the embryo's Hill radius defined by

$$R_{Hill} = a\left(\frac{m_c}{3M_*}\right)^{1/3}.$$

The isolation mass is given by

$$m_{iso} = 2\pi a.2\Delta a_{max}.\Sigma_s \tag{159}$$

where Σ_s is the surface density of solids (planetesimals), a is the semimajor axis of the embryo. Substituting for Δa_{max} gives

$$m_{iso} = 2\pi a.C.a\left(\frac{m_{iso}}{3M_*}\right)^{1/3}.\Sigma_s, \tag{160}$$

and solving for m_{iso} gives

$$m_{iso} = \sqrt{8\pi^3}a^3C^{3/2}\left(\frac{1}{3M_*}\right)^{1/2}\Sigma_s^{3/2}. \tag{161}$$

Detailed computation of the stability of planetesimal orbits perturbed by a planet gives $C = 2\sqrt{3}$. Taking $\Sigma_s = 100$ kg m^{-2} at 1 AU gives $m_{iso} = 0.07$ M$_\oplus$, suggesting that the terrestrial planets in our Solar System did not form by embryos simply accreting planetesimals in their feeding zones. At a distance of 5 AU, using the same solids surface density $\Sigma_s = 100$ kg m^{-1}, we obtain $m_{iso} \simeq 9$ M$_\oplus$. This is in reasonable agreement with the estimated masses of the rock and ice cores at the centres of Saturn, Uranus and Neptune. The situation with Jupiter is more uncertain since here the core mass estimates are more sensitive to assumptions about the equation of state for hydrogen under mega-bar pressures – which is not well-constrained.

7.4 Overview of terrestrial planet formation

We conclude a short discussion about terrestrial planet formation by summarizing briefly the main stages of the process:

1. Dust particles agglomerate to form, eventually, planetesimals. Most likely this occurs via pairwise collisions, though how (or whether) this process can work for cm to meter scale particles remains somewhat murky. There may be a role for gravitational instability.

2. Growth beyond planetesimals occurs via direct collisions, with an increasing role for gravitational focusing as masses become larger. Dynamical friction keeps the velocity dispersion of the most massive bodies low. Gas drag keeps the velocity dispersion of the the planetesimals low. A phase of runaway growth occurs in which a few bodies grow rapidly at the expense of the rest.

3. Runaway growth ceases once the largest bodies become massive enough to stir up the planetesimals in their vicinity. A phase of *oligarchic growth* ensues, in which the largest objects grow more slowly than they did during runaway growth, but still more rapidly than small bodies. Growth continues in this mode until the isolation mass is approached, at which point growth slows further.

4. Further evolution occurs as a result of collisions between the initially relatively isolated planetary embryos left over after oligarchic growth. The embryos are perturbed onto crossing orbits due to the influence of the giant planets and mutual secular resonances. The final assembly of the terrestrial planets takes around 100 Myr, with the predicted configuration varying depending upon the assumed surface density of planetesimals and existence (or not) of giant planets. In the Solar System, the final giant impact on the Earth is widely considered to have given rise to the ejection of enough mass into orbit around the proto-Earth to subsequently form the Moon. This lunar-forming impact has been dated to have arisen at ~ 60 Myrs after the formation of the Sun. This dating has been achieved using the isotopic ratios of ^{186}W and ^{182}W (Tungsten). Early in the history of the Solar system, radiative Hafnium ^{182}Hf decays into ^{182}W with a half-life of 9 Myrs. ^{182}Hf is considered to be a 'lithophile' element, which means that if the Earth becomes completely molten it will remain in the Silicate mantle and crust which forms on cooling. ^{182}W on the other hand is considered to be a 'siderophile' element, meaning that it prefers to follow the heavy element iron as it sinks to the centre of the Earth to form the core. By measuring the amout of of ^{182}W to ^{186}W in the Earth's crust at the present time, it is possible to calculate

the time after the formation of the Sun when the Earth was last molten - which occured during the last lunar-forming giant impact.

The dominant uncertainties in theoretical models for terrestrial planet formation are arguably found during stage 1 — the formation of planetesimals. It is also true that most simulations of the late stages of terrestrial planet formation lead to planetary eccentricities that are slightly larger than those observed in the Solar System. This signals the need for additional dissipation at late epochs, possibly from the dynamical effect of a surviving population of smaller bodies.