3 Lecture 3

3.1 Accretion onto Compact Objects

Accretion of gas onto compact objects can provide a powerful energy source. In theory one can obtain an amount

$$E_m = \frac{GM}{R}$$

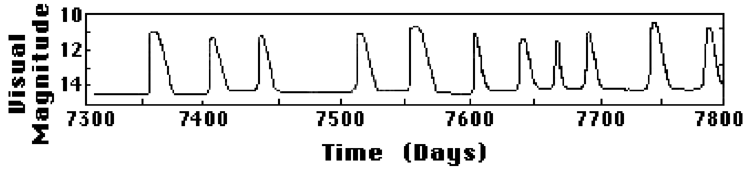
of energy per unit mass by accreting onto a central mass M of radius R.
For neutron stars, $E_m \sim 0.1c^2 \sim 10^{16}$ J kg^{-1}.
For black holes, $E_m \sim 0.1c^2 \sim 10^{16}$ J kg^{-1}.
$\Rightarrow$ 10 percent efficiency of direct mass–energy conversion ($E = mc^2$).
For white dwarfs $\frac{GM}{R} \sim 2.6 \times 10^{13}$ J kg^{-1} (where $M = 1$ M$_\odot$, $R = 5 \times 10^6$ m).
For comparison, nuclear energy release by conversion of H $\rightarrow$ Fe gives $E_m \sim 9 \times 10^{14}$ J kg^{-1}. Thus, accretion onto compact sources can approach or exceed nuclear energy as a power source because compact objects have very deep gravitational potential wells.

Infalling material will generally have some angular momentum, which means that the material will usually go into orbit around the central object if the central object is compact. The infalling material will form an accretion disc. Energy can be dissipated in the accretion disc, in particular through viscosity in the gas, which heats the disc. The hot disc will emit electromagnetic radiation as black body radiation, and this can be powerful enough to be observed from the Earth. The disc is usually too small to be resolved from the Earth, but the total radiation from the disc can be measured and interpreted.

Accretion on to compact objects is observed in a number of circumstances. In *active galactic nuclei*, gas forms an accretion disc around the supermassive black hole in the nucleus of a galaxy. Examples include the nuclei of Seyfert galaxies, radio galaxies, BL Lacertae objects and quasars.

Semi-detached interacting binary stars conventionally have accretion discs, as we found in Chapter 2. The compact companion can be a white dwarf, a neutron star or a black hole. Many interacting binary stars are observed within our Galaxy, usually because of the energy emitted by the accretion discs. Their accretion discs in some cases are so hot that they emit X-rays that can be observed by X-ray satellites (the compact component must be a neutron star or black hole for the disc to be heated this much). The emission from the accretion discs can vary strongly over time as the rate that material moves through the accretion disc changes, and the corresponding changes in brightness can be observed from the Earth. *Cataclysmic variables* (C.V.s) have a white dwarf primary component and a less massive star that fills its Roche lobe: gas spills from the lower mass star into an accretion disc around the white dwarf. (Incidentally, the first C.V. known was discovered London in 1855 – from Regent's Park.)

As an example, consider accretion onto white dwarfs in C.V. (cataclysmic variable) systems. A particular class of C.V. are the dwarf novae. These are typically composed of a 1 M$_\odot$ white dwarf and a 0.5 M$_\odot$ main sequence star, and have orbital periods of a few hours. Mass transfer through the L$_1$ point occurs continuously, but accretion onto



The above figure shows the light curve of the C.V. system SS Aurigae. Notice the regular outbursts, typically lasting a few days, interspersed by periods of quiescence, typically lasting periods of a few weeks or months.

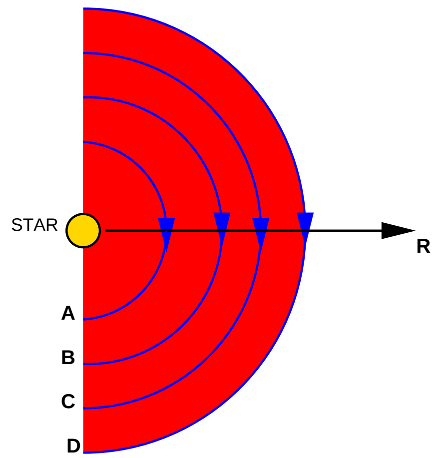
the white dwarf occurs in outbursts. The outbursts occur every month or so, and have a typical decay time of a few days. This is evidence that mass flows through the disc on a time scale of a few days, which should be contrasted with the $\sim$ few hour orbital period at the disc outer edge. The transport of mass through the disc thus occurs on a time scale of $\sim$ 100 orbital periods at the disc outer edge. It is thought that the outbursts are caused by an instability occurring in the disc, causing the viscosity to vary as a function of time, thus inducing a time–dependent flow of matter onto the central white dwarf.

3.2 Angular Momentum Transfer Mechanisms

While the gas in accretion discs moves in nearly circular orbits, there is also a slow drift of material inwards. For this to occur, gas must lose angular momentum. This angular momentum can be exchanged with other gas, or can be lost through torques acting on the gas. Therefore *angular momentum transport* occurs. It is this transport of angular momentum outwards that allows gas to drift inwards, and this transfer of material produces the accretion. Without angular momentum transport, there would be no transfer of gas inwards and therefore no accretion: the gas would remain in stable circular orbits.

In this section we will briefly consider the mechanisms that are thought to be responsible for angular momentum transfer in discs.

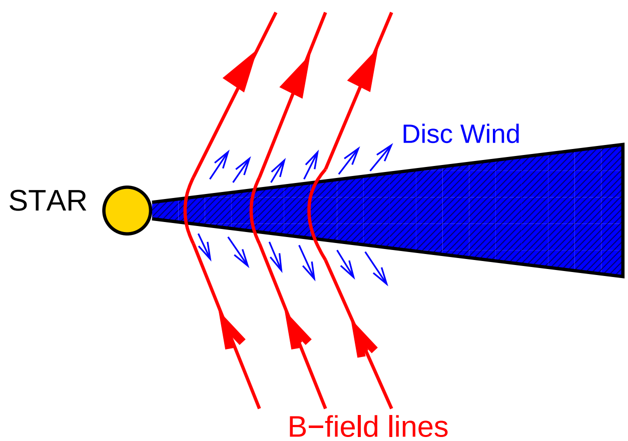
3.2.1 Internal Disc Mechanisms - Viscosity/Friction



In the diagram matter at A rotates faster than at D. Frictional forces tend to slow it down due to interaction with the slower material exterior to it, while matter at D speeds up. The net effect is that A loses angular momentum to D, so A moves inward and D moves outward. Note that specific angular momentum in Keplerian orbit is $j = \sqrt{GMR}$. The frictional effect is produced by viscosity acting in the disc. In order to account for the evolutionary time scales of astrophysical discs the viscosity needs to be large, and cannot be accounted for by the usual molecular viscosity. This has led to the belief that the viscosity is generated by turbulence in the disc.

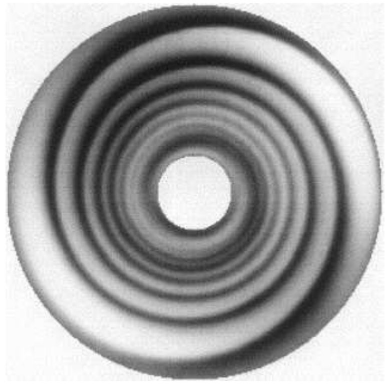
3.2.2 Other Mechanisms

(a) Global Magnetic Fields and Winds



In this scenario, a small amount of mass is lost in a wind that is launched from the surface of the disc, and channelled along field lines. The coupling between the matter leaving in the wind and the disc material itself, *via* the magnetic field lines, leads to a torque being exerted on the disc, and angular momentum being removed. Young stars in the process of forming are often observed to have associated collimated outflows and jets, so this method of allowing accretion onto the central star may be important during star formation.

(b) Wave Transport



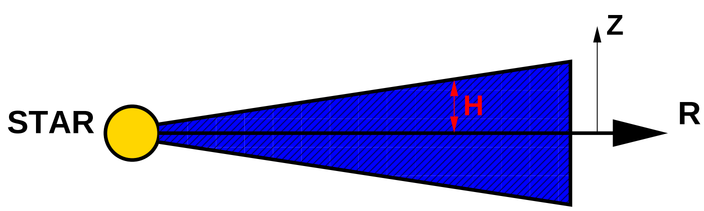
Waves can be excited within a gaseous accretion disc by an external companion and lead to the transport of angular momentum. This process occurs through the tidal interaction of the accretion disc and donor star in close binary systems at the disc edge, and leads to the excitation of spiral density waves and disc truncation.

3.3 Viscous Disc Theory

In describing a viscous accretion disc, we will generally use cylindrical polar coordinates (R, ϕ, z), and will assume that the disc is axisymmetric. We will denote the semi-thickness (half-thickness) of the disc by H, and we will consider discs for which $H(R) \ll R$. We shall also assume Keplerian motion (the mass M of the central star/object dominates). In general, we will work with the surface density, Σ which is defined as

$$\Sigma(R) \;=\; \int_{-\infty}^{\infty} \rho(R)\,\mathrm{d}z \ . \tag{52}$$

3.3.1 Hydrostatic Equilibrium in the Vertical Direction



The equation of hydrostatic equilibrium in the z direction is

$$\frac{1}{\rho}\,\frac{\partial P}{\partial z} \;=\; -\,\frac{GM}{(R^2+z^2)^{3/2}}\,z \ , \tag{53}$$

where P is the gas pressure and ρ the density at the point (R, ϕ, z). For a thin disc (*i.e.* $z \ll R$) this may be approximated as

$$\frac{1}{\rho}\,\frac{\partial P}{\partial z} \;=\; -\,\frac{GMz}{R^3} \;=\; -\,\Omega^2\,z \ , \tag{54}$$

where $\Omega^2 = GM/R^3$. If we assume the gas is isothermal at this distance R from the centre, then $P = \mathcal{R}\rho T/\mu$ is constant over z at this radius R, and equation (54) becomes

$$\frac{1}{\rho}\frac{\mathcal{R}}{\mu}T\frac{\partial\rho}{\partial z} = -\,\Omega^2 z \tag{55}$$

(T is the temperature, μ the molecular mass, and $\mathcal{R}$ the gas constant). Integrating equation (55)

$$\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = -\frac{\Omega^2\mu}{\mathcal{R}T}\int_0^z z\,\mathrm{d}z\ , \tag{56}$$

where ρ_0 is the midplane density (i.e. ρ at $z = 0$ for the particular value of R), which evaluates to

$$\rho(z) = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right) \tag{57}$$

where

$$H^2 = \frac{\mathcal{R}T}{\mu\Omega^2}\ . \tag{58}$$

The *isothermal* speed of sound in an ideal gas of temperature T is

$$c_s = \sqrt{\frac{\mathcal{R}T}{\mu}} \tag{59}$$

where μ is the molecular mass, and $\mathcal{R}$ is the gas constant. This is the speed that sound waves passing through the gas would have if the temperature of the gas did not vary due to the temporary, sudden changes in pressure caused by the sound waves. The *adiabatic* speed of sound, however, is a more realistic measurement of the actual sound speed in the gas, and is given by

$$c_{s\ ad} = \sqrt{\frac{\gamma\mathcal{R}T}{\mu}}$$

where γ is the ratio of specific heat capacities. Note that the differences between the isothermal and adiabatic sound speeds are small because $\sqrt{\gamma} \simeq 1$ (for example, $\gamma = 1.67$ for a monatomic gas, which gives $\sqrt{\gamma} = 1.3$).
Combining equations (58) and (59), we get

$$H^2\,\Omega^2 = c_s^2\ . \tag{60}$$

Here, c_s is the isothermal sound speed. This gives us a very simple, but important, relation between the half-thickness $H(R)$ and angular velocity $\Omega(R)$ at a particular distance R from the central star,

$$\boxed{H\ \Omega = c_s} \tag{61}$$

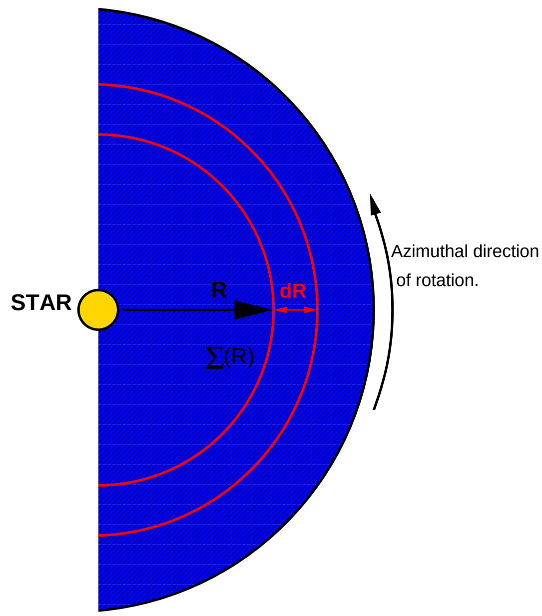
(Remember that this applies to Keplerian discs in which the temperature is constant with z. It applies at each R.)

Calculating the surface density from equation (57) we have

$$\Sigma \;=\; \int_{-\infty}^{\infty} \rho \, \mathrm{d}z \;=\; \int_{-\infty}^{\infty} \rho_0 \exp\left(-\frac{z^2}{2H^2}\right) \mathrm{d}z \;=\; \rho_0 \sqrt{2\pi}\, H \;\;,$$

which relates Σ to ρ_0 and H (this uses the standard integral $\int_0^\infty \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \sigma\sqrt{2\pi}$, because the integral of the Gaussian function cannot be expressed in terms of any simple functions). Note that $H/R = c_s/(R\,\Omega)$, so that we have a thin disc (i.e. $H/R \ll 1$) if the sound speed is small compared to the orbital velocity $v = R\,\Omega$.

3.3.2 Viscous Evolution



In the following analysis, we will use cylindrical polar coordinates (R, ϕ, z) and assume that the fluid variables are independent of ϕ (*i.e.* the disc is axisymmetric).

Consider a ring with its inner edge at radius R and having thickness $\mathrm{d}R$. We are going to consider the viscous forces acting on this ring due to material in the disc lying interior and exterior to it.

For a Newtonian Fluid, the 'viscous stress' is proportional to the velocity gradient, so that the viscous stress is of the general form:

$$\tau \;=\; \eta \frac{\mathrm{d}v}{\mathrm{d}x} \;\;,$$

where η is the coefficient of viscosity. This is the *force per unit area* acting in the fluid due to the frictional effect of the viscosity restricting the deformation of the fluid (*i.e.* that associated with the velocity shear within a differentially rotating disc). In general we will use the kinematic viscosity, denoted as ν, and defined by

$$\nu \;=\; \frac{\eta}{\rho} \;. \tag{62}$$

where ρ is the density of the fluid. From the definition of the viscous stress tensor in the Navier-Stokes equation, it can be shown that the force per unit area acting on the inner edge of the ring is

$$- \nu \rho R \frac{\mathrm{d}\Omega}{\mathrm{d}R} ,$$

where it should be noted that this force is zero for a uniformly rotating fluid, as required. Integrating vertically and azimuthally around the ring gives the total force acting on the inner edge of the ring

$$F_{in} = - 2\pi R^2 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} . \tag{63}$$

This force acts in the direction of rotation and is > 0, so the ring is accelerated. The torque is

$$R F_{in} = - 2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} . \tag{64}$$

The torque acting at the outer edge (slowing the ring down) is

$$RF_{out} = \left[2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right]_{R+\mathrm{d}R} . \tag{65}$$

The net torque acting is the sum of equations (64) and 65)

$$\begin{aligned} \mathcal{T} &= \left[2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right]_{R+\mathrm{d}R} - \left[2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right]_{R} \\ &= \frac{\partial}{\partial R} \left[2\pi R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right] \mathrm{d}R . \end{aligned} \tag{66}$$

But the mass of the ring is $2\pi R \Sigma \, \mathrm{d}R$, so that the torque per unit mass is

$$\mathcal{T}_m = \frac{1}{R\Sigma} \frac{\partial}{\partial R} \left[R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right] . \tag{67}$$

Now the torque per unit mass $\mathcal{T}_m = \mathrm{d}j/\mathrm{d}t$, where $j = R^2 \, \Omega(R)$ is the specific angular momentum. It may be shown that

$$\begin{aligned} \frac{\mathrm{d}j}{\mathrm{d}t} &= \frac{\partial j}{\partial t} + (\mathbf{v} \boldsymbol{.} \boldsymbol{\nabla}) j \quad \text{(the convective derivative)} \\ &= v_R \frac{\partial j}{\partial R} , \end{aligned} \tag{68}$$

in this case since j is a function of R only. Thus we have from equation (67)

$$v_R \frac{\partial}{\partial R} (R^2 \Omega) = \frac{1}{R\Sigma} \frac{\partial}{\partial R} \left[R^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right] , \tag{69}$$

using $j = R^2 \Omega$ for the specific angular momentum. Now we introduce the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \boldsymbol{.} (\rho \mathbf{v}) = 0 , \tag{70}$$

which may be written in cylindrical polar coordinates as

$$\frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\big(R\rho v_R\big) + \frac{\partial}{\partial z}\big(\rho v_z\big) = 0 \tag{71}$$

(where we have neglected the ϕ component because of the axisymmetry). Vertical integration gives

$$\int_{-\infty}^{\infty} \frac{\partial\rho}{\partial t}\,\mathrm{d}z + \int_{-\infty}^{\infty} \frac{1}{R}\frac{\partial}{\partial R}\big(R\rho v_R\big)\,\mathrm{d}z + \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\big(\rho v_z\big)\,\mathrm{d}z = 0 \ .$$

Using equation (52) for Σ in terms of ρ,

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\big(R\Sigma v_R\big) = 0 \ , \tag{72}$$

where we have assumed $v_z = 0$. From equation (69) we can write

$$R\,\Sigma\,v_R = \left[\frac{\partial}{\partial R}\big(R^2\Omega\big)\right]^{-1} \frac{\partial}{\partial R}\left[R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right] \ . \tag{73}$$

Substituting this into equation (72) we obtain

$$\boxed{\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left\{\left[\frac{\partial}{\partial R}\big(R^2\Omega\big)\right]^{-1} \frac{\partial}{\partial R}\left[R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\right\} = 0 \ ,} \tag{74}$$

which is the well-known diffusion equation that governs the time evolution of surface density Σ in a viscous accretion disc at a distance R from the central object.
For the Keplerian case $\Omega = \sqrt{GM/R^3}$ and

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left\{\nu\Sigma R^{1/2}\right\}\right] \ . \tag{75}$$

An important issue is the timescale for changes in the surface density Σ. Equation (75) for the rate of change of surface density, $\partial\Sigma/\partial t$, in a Keplerian disc shows that that it depends only on R, ν and Σ. The timescale, therefore, must depend only on R, ν and Σ.
We can obtain a relation for the timescale τ_{ev} of evolution in Σ using a dimensional argument. R has dimensions of length L, the kinematic viscosity ν has L^2T^{-1}, Σ has ML^{-2}, and the timescale has T. So we expect

$$\tau_{ev} \propto \frac{R^2}{\nu} \ . \tag{76}$$

Detailed theoretical modelling finds that

$$\tau_{ev} \simeq \frac{R^2}{3\nu} \ . \tag{77}$$

Information about the timescale can be used to investigate the cause of the viscosity. The kinematic viscosity ν has dimensions L^2/T or Lv, where L =length, T =time, v = velocity. Diffusion at the molecular level generates viscosity in fluids, and is called molecular viscosity. For molecular viscosity, the appropriate length scale, L, is the mean free path, and the appropriate velocity, v, is the sound speed c_s (since this is the characteristic velocity of molecules in a gas). Molecular viscosity acts because the molecules in a gas have random motions which allow them to diffuse across shearing interfaces in a fluid. The effect of this is to generate friction between adjacent regions of the fluid that are moving relative to one another. It turns out, however, that molecular viscosity is too small to explain the evolutionary time scales of accretion discs, and that turbulence is thought to provide the required stresses in the fluid.

In disc theory we use an 'alpha' model for the kinematic viscosity, where $L = H$, $v = c_s$ and

$$\nu \;=\; \alpha\, c_s\, H \;=\; \alpha\, \Omega^2\, H \;. \tag{78}$$

The interpretation of this expression is that the viscosity is generated by turbulent eddies with characteristic mean free path H and speed $\sim \alpha c_s$.

α is an unknown parameter, and its determination requires a theory of turbulence. Recent computer simulations of MHD (magnetohydrodynamic) turbulence in accretion discs, however, suggest $\alpha \sim 0.01 - 0.1$.

Consider the disc evolution time given by equation (77)

$$\tau_{ev} \;\simeq\; \frac{R^2}{3\nu} \;\simeq\; \frac{R^2}{3\alpha H^2 \Omega} \;=\; \frac{1}{6\pi\alpha}\left(\frac{R}{H}\right)^2 \;\text{ orbits.}$$

For C.V.'s, $H/R \simeq 0.03$, so that $\alpha \sim 0.1$ gives $\tau_{ev} \sim 500$ orbits, which is similar to that required for the explanation of the duration of outbursts of dwarf novae (see earlier).