4 Lecture 4

4.1 Steady State Disc Theory

In the case of a steady state we have $\partial\Sigma/\partial t = 0$ in equations (74) and (75), at all points in the accretion disc.
Mass will flow inwards through the disc at a rate $\dot{m}_d$ (mass per unit time) as angular momentum is transported outwards. Considering mass conservation we have that

$$\dot{m}_d \;=\; -\,2\pi R\Sigma\, v_R \tag{79}$$

which is the rate of mass flow through a circle of radius R, and is a solution to the time-independent continuity equation (72).
From equation (69) we have that

$$v_R\,\frac{\partial j}{\partial R} \;=\; \frac{1}{R\Sigma}\,\frac{\partial}{\partial R}\left[R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right] \tag{80}$$

where we have used $j = R^2\Omega$. Substituting equation (79) into equation (80) we obtain

$$-\,\frac{\dot{m}_d}{2\pi}\,\frac{\partial j}{\partial R} \;=\; \frac{\partial}{\partial R}\left[R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right] \tag{81}$$

Consider the steady–state diffusion equation from (74), with $j = R^2\Omega$,

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\left[\frac{\partial j}{\partial R}\right]^{-1}\frac{\partial}{\partial R}\left[R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right]\right) \;=\; 0\ . \tag{82}$$

Integrating once gives

$$\frac{\partial}{\partial R}\left[R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right] \;=\; C\,\frac{\partial j}{\partial R} \;=\; -\,\frac{\dot{m}_d}{2\pi}\,\frac{\partial j}{\partial R} \tag{83}$$

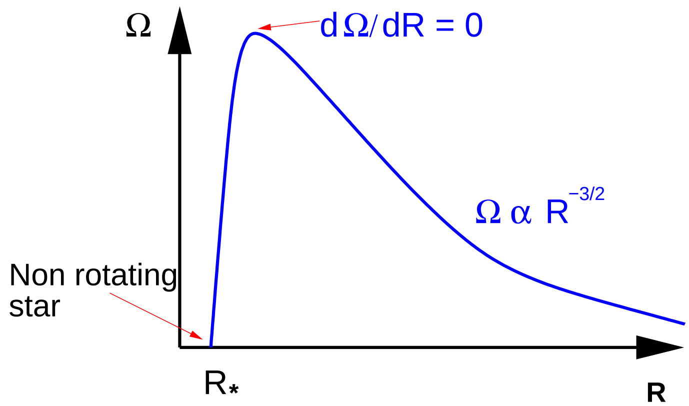
where we have used equation (81) to evaluate the constant of integration, C. Integrating equation (83) gives

$$2\pi R^3\nu\Sigma\,\frac{\mathrm{d}\Omega}{\mathrm{d}R} \;=\; -\,\dot{m}_d\, j \;+\; D\ , \tag{84}$$

where D is a constant of integration.
D is usually calculated by considering the disc close to the star. In particular, we note that if the disc is to join onto a non rotating, or slowly rotating star, then at a position close to the star $\mathrm{d}\Omega/\mathrm{d}R = 0$, such that no viscous stresses act at this point.
If we call R_{max} the value of R where $\mathrm{d}\Omega/\mathrm{d}R = 0$, Equation (84) becomes at this position

$$D \;=\; \dot{m}_d\, R^2_{max}\,\Omega(R_{max})\ . \tag{85}$$

If we assume that the disc is in Keplerian rotation at positions $R \gtrsim R_{max}$ then we have $\Omega(R) = \sqrt{GM_*}R^{-3/2}$, $\mathrm{d}\Omega/\mathrm{d}R = -3\sqrt{GM_*}R^{-5/2}/2$, and $\Omega(R_{max}) = \sqrt{GM_*}R_{max}^{-3/2}$,



where M_* is the mass of the central star and G is the constant of gravitation. We therefore obtain the expression

$$\nu\Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_{max}}{R}\right)^{1/2}\right] \tag{86}$$

However, the value R_{max} of the R coordinate where the angular velocity Ω is a maximum is close to the radius of the star, R_*. So we shall approximate $R_{max} \simeq R_*$. We therefore get

$$D = \dot{m}_d R_*^2 \Omega(R_*) \tag{87}$$

and so

$$\nu\Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{88}$$

by inserting (87) into (84). Convince yourself that this is true by obtaining equation (88).

4.2 Energy Production

We have a steady state solution for $\nu\Sigma$, but we are interested in accretion discs as a source of energy production since ultimately we observe radiation that is emitted from the disc surface. Energy production occurs *via* viscous dissipation. We will now consider the rate of energy production due to viscous dissipation in an accretion disc. The Navier-Stokes equation is

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \tag{89}$$

where

$$\sigma_{ij} = -P\,\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{v}\,\delta_{ij}\right) . \tag{90}$$

σ_{ij} is known as the stress tensor. The first term on the r.h.s. of equation (90) is due to the thermal pressure, and the remaining terms that are proportional to the velocity gradients are due to the viscous stresses acting in the fluid. We have used suffix notation where the subscripts i, j refer to the coordinate directions (e.g. in Cartesian coordinates $v_1 \equiv v_x,\ v_2 \equiv v_y,\ v_3 \equiv v_z$). Each component of σ_{ij} refers to the force per unit area acting in the direction i on a surface in the fluid whose normal points in the j direction.

We will just consider the viscous part of the stress tensor, so that equation (89) becomes

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial \sigma'_{ij}}{\partial x_j} \tag{91}$$

where

$$\sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}\boldsymbol{.}\mathbf{v}\,\delta_{ij}\right) . \tag{92}$$

To get the change in kinetic energy from equation (91) we take the dot product with v_i:

$$v_i\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v_i^2\right) = \frac{1}{\rho}v_i\frac{\partial}{\partial x_j}\sigma'_{ij} \tag{93}$$

from which we obtain

$$\begin{aligned}\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v_i^2\right) &= v_i\frac{\partial}{\partial x_j}\sigma'_{ij}\\ &= \frac{\partial}{\partial x_j}\left(v_i\sigma'_{ij}\right) - \sigma'_{ij}\frac{\partial v_i}{\partial x_j} .\end{aligned} \tag{94}$$

The first term on the r.h.s. of equation (94) is due to the action of viscosity/friction transferring momentum from one region of the fluid to another. There is a corresponding transfer of kinetic energy associated with this momentum transfer. The second term on the r.h.s of equation (94) is the rate of viscous dissipation per unit volume, and gives the rate at which the viscous forces remove kinetic energy from the system and convert it into heat/internal energy.
The rate of viscous dissipation per unit volume may be expressed as

$$\sigma'_{ij}\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\sigma'_{ij}\frac{\partial v_i}{\partial x_j} + \sigma'_{ji}\frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\sigma'_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{95}$$

because σ'_{ij} is symmetric with respect to i and j (see equation (92)).
If we assume that $\boldsymbol{\nabla}\boldsymbol{.}\mathbf{v} = 0$ (as is the case here, as can be seen from the continuity equation (9) for $\rho =$ constant) then we can write

$$\frac{1}{2}\sigma'_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = 2\rho\nu e_{ij}e_{ij} \tag{96}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

is the rate of strain tensor. In writing equation (96) we have used $\eta = \nu\rho$. To obtain the viscous dissipation per unit area we must integrate equation (96) vertically through the disc

$$\epsilon_D = \int 2\,\rho\,\nu\,e_{ij}\,e_{ij}\,\mathrm{d}z = R^2\nu\,\Sigma\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 \tag{97}$$

where we have evaluated e_{ij} for the velocity field $\mathbf{v} = v_\phi \hat{\mathbf{e}}_\phi = R\,\Omega\,\hat{\mathbf{e}}_\phi$, where $\hat{\mathbf{e}}_\phi$ is the unit vector in the ϕ direction. For a Keplerian disc in a steady state $\mathrm{d}\Omega/\mathrm{d}R = -(3\Omega)/(2R)$, giving

$$\epsilon_D = \frac{9}{4} \nu \, \Sigma \, \Omega^2 \ . \tag{98}$$

We have now seen that the rate of heat generation in the disc due to viscous dissipation of kinetic energy is given by

$$\epsilon_D = \frac{9}{4} \Omega^2 \, \nu \, \Sigma \ . \tag{99}$$

Combining (99) and equation (88) we obtain the expression

$$\epsilon_D = \frac{9}{4} \nu \, \Sigma \, \Omega^2 = \frac{3}{4\pi} \dot{m}_d \left[1 - \left(\frac{R_*}{R} \right)^{1/2} \right] \frac{GM}{R^3}. \tag{100}$$

The flux of radiation from an optically thick disc radiating as a black body is given by

$$F = 2\sigma T_{eff}^4 \tag{101}$$

where the factor of 2 arises because the disc radiates from two sides. In a steady-state, the energy dissipated = energy radiated, so equating expression (101) and (100) gives an equation for the effective temperature as a function of radius

$$T_{eff} = \left[\frac{3GM\dot{m}_d}{8\pi\sigma R^3} \left(1 - \left\{ \frac{R_*}{R} \right\}^{1/2} \right) \right]^{1/4} \tag{102}$$

Note that this expression is independent of the kinematic viscosity ν, indicating that the temperature in an accretion disc is not sensitive to the physical process that generates the viscosity, but only on the rate of mass flow through the disc that arises because of it.
To make estimates of the temperature in accretion discs, we will ignore R_* (since it is usually small), so that

$$T_{eff} \simeq \left[\frac{3GM\dot{m}_d}{8\pi\sigma R^3} \right]^{1/4} \ . \tag{103}$$

Numerically we can write

$$T_{eff} = 6.6 \times 10^4 \,\mathrm{K} \left(\frac{\dot{m}}{10^{-9} M_\odot/\mathrm{yr}} \right)^{1/4} \left(\frac{M}{M_\odot} \right)^{1/4} \left(\frac{R}{10^7 \mathrm{m}} \right)^{-3/4} \ . \tag{104}$$

Apply to C.V.s:
For $R \sim 10^7$ m, $M \sim 1$ $\mathrm{M}_\odot$, $\dot{m} \sim 10^{-9}$ $\mathrm{M}_\odot$ yr^{-1}, we obtain $T \sim 6 \times 10^4$ K, which is typical of the values that apply to C.V.s. Most of the energy in this case is emitted in the U.V.

Apply to Neutron Stars or Black Holes:
Here we take $R \sim 10^4$ m. Note, however, in this case the accretion rate is limited by the Eddington limit. During the process of accreting gas onto compact objects, the

gas may become very hot and radiation pressure will then be important. For mass accretion rates, $\dot{m}$, above a critical value (known as the Eddington limit), radiation pressure dominates gravity, preventing further accretion. To obtain a simple estimate of the Eddington limited accretion rate, consider the competition between radiation pressure and gravity in a spherical cloud:

$$-\frac{GM}{r^2} - \frac{1}{\rho}\frac{\mathrm{d}P_{rad}}{\mathrm{d}r} > 0 \tag{105}$$

if radiation pressure beats gravity, where $P_{rad} = aT^4/3$. The luminosity is given by $L = 4\pi r^2 F$ where the radiative flux, F, is given by

$$F = -\frac{4ac}{3\kappa\rho}T^3\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{c}{\kappa\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{aT^4}{3}\right) . \tag{106}$$

Thus we can write the luminosity as

$$L(r) = -\frac{4\pi r^2 c}{\kappa\rho}\frac{\mathrm{d}P_{rad}}{\mathrm{d}r} \tag{107}$$

and equation (105) becomes

$$-\frac{GM}{r^2} + \frac{L\kappa}{4\pi c r^2} > 0 \tag{108}$$

from which we get

$$\kappa L > 4\pi c\, GM . \tag{109}$$

The luminosity generated by mass accretion at a rate $\dot{m}$, onto an object of mass M and radius R is given by

$$L = \frac{GM\dot{m}}{R} . \tag{110}$$

If we take the radius $R \sim 5R_S$, where $R_S = 2GM/c^2$ is the Schwarzschild radius, then

$$L = 0.1\,\dot{m}c^2$$

which combined with (109) gives

$$\dot{m} = \frac{4\pi cGM}{0.1c^2\kappa} = \frac{40\pi GM}{c\kappa} .$$

For electron scattering opacity (which is dominant in a hot ionised plasma), $\kappa \simeq 0.04\ \mathrm{m^2\,kg^{-1}}$ from electrons in ionised hydrogen giving

$$\dot{m} = 2\times 10^{-8}\,M\ \mathrm{yr}^{-1} .$$

Taking this value, and $R = 10^4$ m in equation (104) gives $T_{eff} \simeq 2.5\times 10^7$ K for $M = 1M_\odot$. Gas this hot radiates blackbody radiation of very short wavelength.
$\Rightarrow$ Expect X-rays to be emitted.

Apply to Active Galactic Nuclei (AGN):
Take a massive black hole containing $\sim 10^8$ $M_\odot$, with $R = 10^{12}$ m (Schwarzschild radius). Here we get $T_{eff} \simeq 10^5$ K, which is too low to explain the observed X-rays. In this example, the model of a steady accretion disc with thermal emission does not provide an accurate description of an AGN. It is probable that in this case much of the energy is deposited in an optically thin corona above the disc from where it is radiated (cf. the Sun's corona).

Apply to Protostellar Discs:
Here we take $\dot{m} \simeq 10^{-8}$ $M_\odot$ yr^{-1}, and $R \simeq 10^{11}$ m (~ 1 AU). This gives $T_{eff} \simeq 100K$, so that we expect most of emission from protostellar discs to be emitted in the infra-red (as observed).
Thus, we have shown that astrophysical discs display a broad range of temperature environments, ranging between 10^2 and 10^7 K.