5 Lecture 6

5.1 Spectrum of Optically Thick Accretion Disc

Here we are going to calculate how the emitted power (luminosity) radiated from an optically thick disc varies as a function of the emitted frequency, ν. Note that in the following discussion, the symbol ν is used to denote the frequency of the emitted radiation, and not the kinematic viscosity.
From equation (103) we have that the effective temperature at a point at a distance R from the central object in an accretion disc is

$$T_{eff} = \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\right]^{1/4} , \tag{111}$$

where M is the mass of the central object, $\dot{m}$ is the accretion rate on to the central object, G is the constant of gravity ($G = 6.673 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$), and σ is the Stefan constant ($\sigma = 5.670 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$). This equation assumes that the disc is Keplerian, it is in a steady state, angular momentum transport occurs due to viscosity, and energy is liberated due to heating through viscous dissipation. It applies for $R \gg R_*$, the radius of the central object. We can write this as

$$T_{eff} = \beta R^{-3/4} \quad \text{where} \quad \beta \equiv \left[\frac{3GM\dot{m}}{8\pi\sigma}\right]^{1/4} \tag{112}$$

is a constant for a particular disc.
From the Planck function we have that the flux emitted at a particular frequency ν from a blackbody at a temperature T_{eff} is

$$F_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/kT_{eff}\right) - 1} , \tag{113}$$

where c is the velocity of light ($c = 2.9979 \times 10^{-8}\ \mathrm{m\,s^{-1}}$), h is the Planck constant ($h = 6.626 \times 10^{-34}\ \mathrm{J\,s^{-1}}$), and k is the Boltzmann constant ($k = 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$). This is written as F_ν because it is the energy emitted per unit time per unit area of the blackbody *per unit frequency interval*.
The power output from one side of the disc is

$$\begin{aligned}\frac{1}{2}L_\nu &= \int_{R_{in}}^{R_{out}} 2\pi R F_\nu \,\mathrm{d}R \\ &= \left(\frac{2\pi}{c}\right)^2 h \int_{R_{in}}^{R_{out}} \frac{\nu^3 R}{\exp\left(h\nu/kT_{eff}\right) - 1}\,\mathrm{d}R ,\end{aligned} \tag{114}$$

defining L_ν to be the total luminosity per unit frequency interval from both sides of the disc. Note that $T_{eff} \equiv T_{eff}(R)$.
We now make a change of variable

$$x \equiv \frac{h\nu}{kT_{eff}}$$

so that

$$\mathrm{d}x \;=\; \mathrm{d}\left(\frac{h\nu}{kT_{eff}}\right) \;=\; \frac{\mathrm{d}}{\mathrm{d}T_{eff}}\left(\frac{h\nu}{kT_{eff}}\right)\frac{\mathrm{d}T_{eff}}{\mathrm{d}R}\,\mathrm{d}R \;=\; -\,\frac{h\nu}{kT_{eff}^2}\,\frac{\mathrm{d}T_{eff}}{\mathrm{d}R}\,\mathrm{d}R\,.$$

Note that the limits of integration now change so that we use

$$x_{in} = \frac{h\nu}{kT_{eff}(R_{in})}, \quad x_{out} = \frac{h\nu}{kT_{eff}(R_{out})} \;.$$

We will consider frequencies such that $kT_{eff}(R_{out}) \ll h\nu \ll kT_{eff}(R_{in})$, so that $x_{in} \longrightarrow 0$ and $x_{out} \longrightarrow \infty$. In other words, we will consider frequencies that are characteristically emitted from regions of the disc that are not at the hottest inner part or coolest outer part. We can now write equation (114) in the form

$$\frac{1}{2}\,L_\nu \;=\; \left(\frac{2\pi}{c}\right)^2 h\int_0^\infty -\,\frac{kT_{eff}^2}{h}\,\frac{R\nu^2}{\exp{(x)}-1}\,\frac{\mathrm{d}R}{\mathrm{d}T_{eff}}\,\mathrm{d}x \;. \tag{115}$$

We now write

$$-\,\frac{T_{eff}}{R}\,\frac{\mathrm{d}R}{\mathrm{d}T_{eff}} \;=\; -\,\frac{\mathrm{d}\,\ln R}{\mathrm{d}\ln T_{eff}} \;.$$

Taking the logarithm of equation (112) (i.e. $\ln T_{eff} = \ln(\beta R^{-3/4})$) and differentiating,

$$-\,\frac{\mathrm{d}\ln R}{\mathrm{d}\ln T_{eff}} \;=\; \frac{4}{3} \;.$$

Equation (114) may now be written as

$$\frac{1}{2}L_\nu \;=\; \frac{4}{3}\left(\frac{2\pi}{c}\right)^2\int_0^\infty \frac{kT_{eff}R^2\nu^2}{(\exp x - 1)}\,\mathrm{d}x \;. \tag{116}$$

If we combine the expressions

$$x \;=\; \frac{h\nu}{kT_{eff}} \quad \text{and} \quad T_{eff} \;=\; \beta R^{-3/4}$$

we can write

$$T_{eff}\,R^2 \;=\; \beta^{8/3}\left(\frac{kx}{h\nu}\right)^{5/3} \;.$$

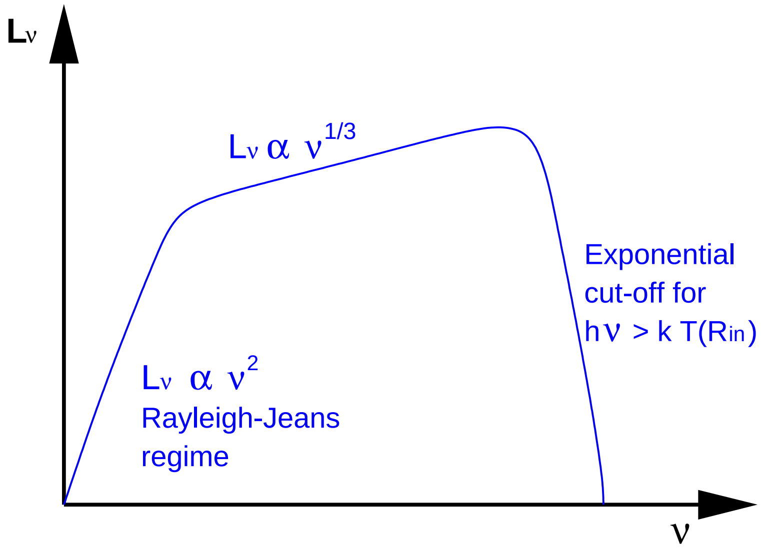
Substituting this into equation (116) yields

$$\frac{1}{2}\,L_\nu \;=\; \frac{16\pi^2}{3c^2}\,(\beta k)^{8/3}h^{-5/3}\,\nu^{1/3}\int_0^\infty \frac{x^{5/3}}{(\exp x - 1)}\,\mathrm{d}x \tag{117}$$

which shows that the middle part of the spectrum of an optically thick accretion disc is expected to show

$$L_\nu \;\propto\; \nu^{1/3} \;.$$

The diagram shows the expected change of L_ν with ν, including the behaviour at both large and small values of ν that we neglected from the above discussion.



5.2 The Boundary Layer

Here we will consider the behaviour of the disc in the vicinity of the central star. The region where the disc joins onto the central star is known as the boundary layer.
In a steady state disc at a distance R from the centre we have from equation (102)

$$\sigma T_{eff}^4 \;=\; \frac{3GM\dot{m}}{8\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right] \;. \tag{118}$$

The total power radiated (the luminosity) is obtained by integrating the flux defined in equation (118) over both surfaces of the disc giving

$$L \;=\; 2\int_{R_{min}}^{\infty} \sigma T_{eff}^4 \,.\, 2\pi R\,\mathrm{d}R$$

assuming for convenience here that the disc extends from $R = R_{min}$ to ∞. (This is the total energy emitted by the disc over all frequencies and over all the area of the disc.) Therefore,

$$L \;=\; 4\pi\int_{R_{min}}^{\infty} \sigma T_{eff}^4\, R\,\mathrm{d}R \;=\; \frac{3}{2}GM\dot{m}\int_{R_{min}}^{\infty}\left[\frac{1}{R^2}-\frac{R_*^{1/2}}{R^{5/2}}\right]\mathrm{d}R \;. \tag{119}$$

This evaluates to

$$L \;=\; \frac{3GM\dot{m}}{2}\left[\frac{1}{R_{min}}-\frac{2R_*^{1/2}}{3R_{min}^{3/2}}\right] \;. \tag{120}$$

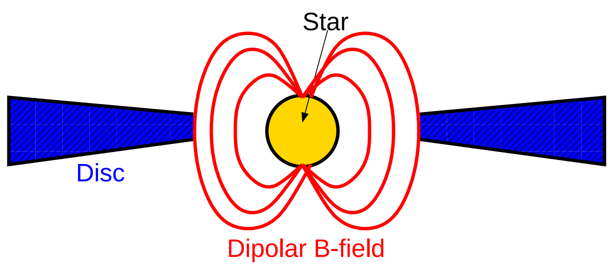
If $R_{min} = R_*$, the radius of the star, this becomes

$$L \;=\; \frac{GM\dot{m}}{2R_*} \;. \tag{121}$$

Note that $GM\dot{m}/(2R_*)$ is the power radiated in going from a circular orbit at ∞ to a circular orbit at radius R_*. This is only half of the available energy, and the other half is stored as the kinetic energy of circular motion at radius R_*. If this material is finally brought to rest, then an additional power of $GM\dot{m}/(2R_*)$ is available if the destruction of the final rotational energy can occur in the boundary layer at radius R_*, assuming that the central star is non-rotating. This could provide an additional source of energy, such as the production of X-rays observed in C.V.s, or the production of U.V. photons in T Tauri systems.

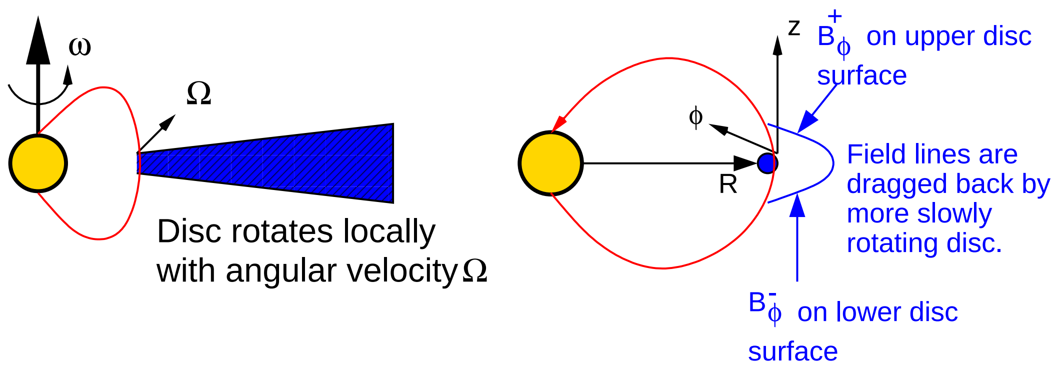
The liberation of this additional energy depends on the existence of a boundary layer close to the star. However, it is possible that this boundary layer does not exist if the disc is disrupted by a strong stellar magnetic field before the star is reached.

5.3 Disc-Magnetosphere Interaction



Suppose that the central star has a strong magnetic field that threads through the disc out to some radius. Further suppose that the star is rotating with angular velocity ω, so that the frozen–in magnetic field also rotates at this rate. The field interacts with the disc, and the more slowly rotating parts of the disc receive an outward torque such that they are repelled from the central star.

Suppose that the star initially has an axisymmetric dipole field. If $\omega > \Omega$, then poloidal magnetic field is converted into toroidal field, and a component of the field in the azimuthal direction, B_ϕ, is produced. This arises because the more slowly rotating disc drags the field lines back so that they become distorted.



5.3.1 Estimate of Torque

The magnetic force per unit volume acting on the disc is given by equation (6)

$$\mathbf{f} = \mathbf{j} \times \mathbf{B} \ . \tag{122}$$

We need to take the ϕ component f_ϕ in order to calculate the torque. In cylindrical coordinates (R, ϕ, z), the component of $\mathbf{j} \times \mathbf{B}$ in the ϕ direction is

$$f_\phi \; = \; (\mathbf{j} \times \mathbf{B})_\phi \; = \; j_z B_R - j_R B_z \ , \tag{123}$$

using the result in Appendix B2. Using Ampère's Law (equation [7]) we obtain

$$\begin{aligned} \mu_0 \, j_R \; &= \; \frac{1}{R} \frac{\partial B_z}{\partial \phi} \; - \; \frac{\partial B_\phi}{\partial z} \\ \mu_0 \, j_z \; &= \; \frac{1}{R} \frac{\partial}{\partial R}(R B_\phi) \; - \; \frac{1}{R} \frac{\partial B_R}{\partial \phi} \ , \end{aligned} \tag{124}$$

and assuming that the field components are independent of ϕ, equation (123) becomes

$$f_\phi \; = \; (\mathbf{j} \times \mathbf{B})_\phi \; = \; \frac{B_R}{\mu_0 R} \frac{\partial}{\partial R}(R B_\phi) \; + \; \frac{B_z}{\mu_0} \frac{\partial B_\phi}{\partial z} \ . \tag{125}$$

The torque per unit volume at a distance R from the centre is given by

$$\mathcal{T}_V \; = \; R \, (\mathbf{j} \times \mathbf{B})_\phi \; = \; \frac{B_R}{\mu_0} \frac{\partial}{\partial R}(R B_\phi) \; + \; \frac{R B_z}{\mu_0} \frac{\partial B_\phi}{\partial z} \ . \tag{126}$$

For a thin disc, $B_R \ll B_\phi, \ B_z$, so we make the approximation of neglecting the first term in equation (126). To get the total torque, $\dot{J}$, we integrate over the disc volume to get

$$\dot{J} \; = \; \int_{-H}^{H} \int_{R_{min}}^{\infty} \mathcal{T}_V \, . \, 2\pi R \, \mathrm{d}R \, \mathrm{d}z \; = \; \int_{-H}^{H} \int_{R_{min}}^{\infty} \frac{R B_z}{\mu_0} \frac{\partial B_\phi}{\partial z} \, 2\pi R \, \mathrm{d}R \, \mathrm{d}z \ , \tag{127}$$

where R_{min} is the radius of the inner edge of the disc. Note that the total torque is equal to the rate of change of the total angular momentum $\dot{J}$ from Newton's second law. We now consider the vertical integration through the disc equation (127). We will make the approximation that $B_z \simeq$ constant through the disc. We will denote the $B_\phi = B_\phi^+$ on the upper face of the disc and $B_\phi = B_\phi^-$ on the lower face of the disc. From the diagram showing the distortion of the field lines, we can see that $B_\phi^+ = -B_\phi^-$. The vertical integration in equation (127) then yields

$$\dot{J} \; = \; 4\pi \int_{R_{min}}^{\infty} \frac{B_z B_\phi^+}{\mu_0} \, R^2 \, \mathrm{d}R \ . \tag{128}$$

In order to make a simple estimate, we will assume that $B_\phi^+ \simeq B_z$, and also that the magnetic field is dipolar so that

$$B_z(R) \; = \; B_z(R_*) \left(\frac{R_*}{R} \right)^3$$

where R_* is the radius of the central star. The integral in equation (128) then evaluates to

$$\dot{J} \; = \; \frac{4\pi}{3} \frac{B_z^2(R_*)}{\mu_0} \frac{R_*^6}{R_{min}^3} \ . \tag{129}$$

We estimate that the inner edge of the accretion disc occurs when the magnetic and viscous torques balance. Taking the viscous torque acting on the inner edge of the disc from equation (64) we obtain

$$\dot{J} \;=\; -\left(2\pi R^3 \nu \Sigma \,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_{min}} \;=\; \frac{4\pi}{3}\,\frac{B_z^2(R_*)}{\mu_0}\,\frac{R_*^6}{R_{min}^3}\;. \tag{130}$$

But from equation (88) we have that $\dot{m} \simeq 3\pi\nu\Sigma$ so that for a Keplerian disc equation (130) becomes

$$\dot{m}\,R_{min}^2\,\Omega(R_{min}) \;=\; \frac{4\pi}{3}\,\frac{B_z^2(R_*)}{\mu_0}\,\frac{R_*^6}{R_{min}^3}\;. \tag{131}$$

Substituting $\Omega = \sqrt{GM/R^3}$ into equation (131) leads to the following expression for the inner radius of the disc

$$\frac{R_{min}}{R_*} \;=\; \left(\frac{4\pi\,B_z^2(R_*)\,R_*^{5/2}}{3\sqrt{GM}\,\dot{m}\,\mu_0}\right)^{2/7}\;, \tag{132}$$

which can be written in terms of physical constants as

$$\frac{R_{min}}{R_*} \;=\; 50\left(\frac{B_z(R_*)}{1\ \mathrm{Tesla}}\right)^{4/7}\left(\frac{R_*}{R_\odot}\right)^{5/7}\left(\frac{\dot{m}}{10^{-8}M_\odot \mathrm{yr}^{-1}}\right)^{-2/7}\left(\frac{M}{M_\odot}\right)^{-1/7}\;. \tag{133}$$

Thus, for a T Tauri star with $\dot{m} = 10^{-8}$ $\mathrm{M}_\odot$ yr^{-1},

$R_{min} \simeq 50$ R_* if $B_z(R_*) = 1$ Tesla

$R_{min} \simeq 13$ R_* if $B_z(R_*) = 0.1$ Tesla

Thus R_{min} is in the range 0.05 to 0.2 AU. Note that R_{min} decreases with increasing $\dot{m}$ and decreasing $B_z(R_*)$.

5.3.2 Equilibrium Rotation of Star

This occurs when $\Omega(R_{min}) = \omega$, the disc rotation period at the inner radius is the same as that of the star.

Why do we get equilibrium ?

If $\Omega(R_{min}) > \omega$ then the star will spin up.

If $\Omega(R_{min}) < \omega$ then the star will spin down.

The situation adjusts until equilibrium is attained with $\Omega(R_{min}) = \omega$.

Consider a neutron star with $M = 1$ $\mathrm{M}_\odot$ accreting at a rate $\dot{m} = 10^{-8}$ $\mathrm{M}_\odot$ yr^{-1}, and with radius $R_* = 1/(7 \times 10^4)$ $\mathrm{R}_\odot$.

For a magnetic field flux density of $B_z(R_*) = 10^8$ Tesla, we get $R_{min} = 647$ R_*. This corresponds to an equilibrium period of 9 seconds, applicable to very young pulsars.

How to get millisecond pulsars ?

$\Rightarrow$ reduce $B_z(R_*)$ to 10^2 Tesla. This reduces R_{min} and gives an equilibrium period of ~ 3.36 milliseconds.