Queen Mary
University of London

M.Sc. EXAMINATION BY COURSE UNIT

ASTM116 Astrophysical Plasmas

8 May 2006 14:30 – 17:30

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination. Numerical answers where required may be determined approximately, to within factors ~ 5, or left in terms of trigonometric or other transcendental functions.

You may quote the following results unless the question specifically asks you to derive it. All notation is standard. Vectors are denoted by boldface type, e.g., $\mathbf{A}$, *while scalars, including the magnitude of a vector, are in italics, e.g.,* $|\mathbf{E}| = E$.

(i) *The Lorentz force on a particle of charge q moving in electric and magnetic fields* $\mathbf{E}$ *and* $\mathbf{B}$ *respectively is given by*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(ii) *Maxwell's Equations*

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{E} &= \frac{\rho_q}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

where $\mu_0 \varepsilon_0 = 1/c^2$.

(iii) *The electric and magnetic fields* $\mathbf{E}$ *and* $\mathbf{B}$ *as measured in a laboratory frame are related to the fields* $\mathbf{E}'$ *and* $\mathbf{B}'$ *measured in a frame moving relative to the laboraty frame at a velocity* $\mathbf{u}$ *by the transformation laws*

$$\begin{aligned}
\mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\
\mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\
\mathbf{E}'_{\perp} &= \frac{\mathbf{E}_{\perp} + \mathbf{u} \times \mathbf{B}}{\sqrt{1-(u^2/c^2)}}
\end{aligned}$$

$$\mathbf{B}'_\perp = \frac{\mathbf{B}_\perp - \dfrac{\mathbf{u}\times\mathbf{E}}{c^2}}{\sqrt{1-(u^2/c^2)}}$$

(iv) The MHD equations for a plasma with electrical conductivity σ:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) &= 0 \\
\rho\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} &= -\nabla p + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} \\
\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\left(p\rho^{-\gamma}\right) &= 0 \\
\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times(\mathbf{V}\times\mathbf{B}) + \frac{1}{\mu_0\sigma}\nabla^2\mathbf{B} \\
\mathbf{E} + \mathbf{V}\times\mathbf{B} &= \mathbf{j}/\sigma
\end{aligned}$$

(vi) The following vector identities and relations

$$\begin{aligned}
\nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\nabla\cdot\mathbf{b}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - \mathbf{b}(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{b} \\
(\nabla\times\mathbf{B})\times\mathbf{B} &= (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2}\right) \\
\nabla\times(\nabla\times\mathbf{B}) &= \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} \\
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\
(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}
\end{aligned}$$

(vii) The following numerical values of physical constants and parameter values:

Name	symbol	value
Electronic Charge	e	1.6×10^{-19} C
Electron volt	eV	1.6×10^{-19} Joules
Electron mass	m_e	9.1×10^{-31} kg
Proton mass	m_p	1.67×10^{-27} kg
Permeability of free space	μ_0	$4\pi\times10^{-7}$ Henry/m
Permittivity of free space	ε_0	8.85×10^{-12} Farad/m
Speed of light in vacuo	c	3×10^{8} m/s
Earth Radius	R_e	6371 km
Astronomical Unit	AU	1.5×10^{11} m
Solar Radius	$R_\odot$	6.96×10^{8}m

SECTION A

Each question carries 10 marks. You should attempt ALL questions.

1. Explain what is meant by "magnetic reconnection," and why it is important. Draw a diagram of the Sweet-Parker model of reconnection. Include on your diagram indications of the magnetic field and bulk velocity vectors in all regions of space, and the location of the diffusion region. What disadvantage does this model have for explaining energy release in astrophysical plasmas?

2. A particle of mass m and charge q moves non-relativistically in static, uniform electric and magnetic fields $\mathbf{E} \equiv E_o\hat{y}$ and $\mathbf{B} \equiv B_o\hat{z}$. By solving the equation of motion for the particle velocity, or otherwise, show that the motion consists of a constant z-velocity, cyclotron motion around $\mathbf{B}$, and a uniform drift. What direction is the drift and how does it depend on the sign of the charge q? Explain qualitatively how curvature of the magnetic field lines, in the absence of an electric field, can lead to a uniform drift.

3. Derive the MHD induction equation for $\partial B/\partial t$ for a non-relativistic plasma. Define the Magnetic Reynolds number R_m. Explain how it is related to the MHD induction equation and describe the limiting behaviour of plasmas with small and large values of R_m.

4. Some comets have two tails: a dust tail and a gas tail. What evidence can be deduced from the gas tail for the existence of a supersonic solar wind? Explain what is meant by MHD "flux freezing" and illustrate how it can be used to explain the behaviour of the magnetic field lines in the vicinity of a comet. Illustrate your answer with a sketch.

5. Define the magnetic moment μ of a charged particle; give both the standard definition and the definition in terms of the particle pitch angle α and kinetic energy W. What important property does μ have, and under what conditions?

A spacecraft above the Earth's magnetic pole fires charged particles of energy 5 keV towards the ionosphere. The magnetic field above the pole can be approximated by $B \approx 10^7/r^3$, where r is distance from the centre of Earth. The spacecraft is at a distance of $r = 21,000$ km.

What approximate range of pitch angles should the particles be released at in order to reach the top of the ionosphere at $r = 7000$ km?

[Assume the particle energy is constant. Use the small angle approximation $\sin\alpha \approx \alpha$, for α in radians.]

SECTION B

Each question carries 25 marks. You may attempt all questions but only marks for the best TWO questions will be counted.

1. Consider a perpendicular, time-steady shock, in a frame in which the shock is stationary in the plane $x = 0$, with an upstream flow velocity $\mathbf{V} = -V\widehat{\mathbf{x}}$, and a uniform upstream magnetic field $\mathbf{B} = B\widehat{\mathbf{z}}$. The normal to the shock surface points upstream $\widehat{\mathbf{n}} = \widehat{\mathbf{x}}$, so that the upstream region has $x > 0$.

(a) [*2 marks*] Assuming ideal MHD, what is the upstream electric field $\mathbf{E}$? Give your answer in component form.

(b) [*14 marks*] A particle of mass m and charge q hits the shock with exactly the upstream flow velocity, and is then reflected specularly, i.e., it reverses its component of velocity normal to the shock.

From the particle's equation of motion, obtain an analytic solution for the velocity $\mathbf{u}$ and position $\mathbf{x}$ in component form, assuming an initial position of $(0,0,0)$. Assume that the magnetic and electric fields are uniform throughout the particle's motion. Use the definition $\Omega = qB/m$.

(c) [*5 marks*] Describe briefly with a sketch the particle's motion after reflection. Derive the following expression for the maximum distance that a reflected particle reaches upstream:

$$x_{\max} = \frac{V}{\Omega}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

(d) [*4 marks*] Describe briefly the overall magnetic field structure of a high Mach number perpendicular shock as observed in space. What is the role of reflected ions at such shocks, and how do they influence the structure?

2. For a uniform static plasma obeying the ideal MHD equations, consider the propagation of small amplitude waves with wave vector $\mathbf{k}$.

The background plasma has mass density ρ_0, magnetic field $\mathbf{B}_0 = B_0\hat{z}$, pressure p_0, and sound speed given by $c_s^2 = \gamma p_0/\rho_0$.

(a) [*5 marks*] By substituting $\mathbf{V} = \mathbf{V}_1$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, etc., (where quantities, subscript 1, are linear perturbations on the equilibrium state, subscript 0) show that the linearized MHD equations take the form:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}_1 = 0 \quad (1)$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \quad (2)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{V}_1 \times \mathbf{B}_0) \tag{3}$$

$$p_1 = c_s^2 \rho_1 \tag{4}$$

(b) [*10 marks*] Assuming plane wave solutions of the form $Q_1 = \delta Q e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$, find the equation for the perturbed quantity $\delta\mathbf{V}$ in terms of the wave properties $\mathbf{k}$ and ω and the background quantities.

(c) [*5 marks*] For waves propagating perpendicular to the background magnetic field, i.e., with $\mathbf{k} \perp \mathbf{B}_0$, show that there exists a wave with $\delta V_z = 0$ and $\delta\mathbf{V} \parallel \mathbf{k}$, which has the dispersion relation

$$\omega^2 = k^2(c_s^2 + v_A^2)$$

where $v_A^2 = B^2/(\mu_0\rho_0)$.

(d) [*5 marks*] For this wave mode, show that variations in both the total magnetic field B and density ρ are compressive, and that they are in phase.

3. A simple model of the solar wind can be derived assuming that the solar corona, described as an isothermal gas obeying the ideal gas pressure law $p = 2nk_BT$, undergoes a steady-state, spherically symmetric and purely radial expansion. The mass density and number density are related by $\rho = nm$, where m is the mean mass of a particle. Neglect the magnetic field.

(a) [*12 marks*] Show that the outflow speed V, as a function of radius r, is governed by:

$$\left(V^2 - \frac{2k_BT}{m}\right)\frac{1}{V}\frac{dV}{dr} = \frac{4k_BT}{mr} - \frac{GM_\odot}{r^2}$$

Ensure that you state clearly any additional assumptions you make.

(b) [*3 marks*] Verify that the following is a solution of this equation:

$$\frac{1}{2}V^2 - \frac{2k_BT}{m}\ln V = \frac{4k_BT}{m}\ln r + \frac{GM_\odot}{r} + K$$

where K is constant.

(c) [*5 marks*] The solar wind solution has an outflow speed V_c at the critical radius r_c such that both sides of the first equation are zero. Determine r_c and V_c, and comment on the special property of the outflow speed at this point.

(d) [*5 marks*] Use your values of r_c and V_c to determine K. Hence obtain an approximate form of the solar wind solution at large heliocentric distance where $r \gg r_c$ and $V \gg V_c$. Explain why the behaviour of this solution as r increases is unphysical.

4. Consider a one-dimensional, steady state, shock at which the x axis is taken parallel to the unit shock normal vector $\widehat{\mathbf{n}}$, such that all quantities depend only on x. The jump across the shock for any quantity X is written as $[X] = X_u - X_d$, where the subscripts u and d refer to the upstream and downstream values, respectively. For a time-steady, one-dimensional shock, the condition $\partial X/\partial x = 0$ implies $[X] = 0$.

The magnetic field and velocity vectors can be split into components normal and parallel to the shock surface. For example,

$$\mathbf{B} = B_x\widehat{\mathbf{n}} + \mathbf{B}_t,$$

where $\mathbf{B}_t$ is the transverse component vector, and B_x is the normal component.

(a) [*4 marks*] Using Maxwell's equations and the one-fluid MHD equations, show that

$$\begin{aligned} [B_x] &= 0 \\ [\rho V_x] &= 0 \end{aligned}$$

(b) [*8 marks*] From the tranverse component of the MHD momentum equation (without gravity), and using the results above, deduce the jump condition:

$$\left[\rho V_x \mathbf{V}_t - \frac{B_x}{\mu_0}\mathbf{B}_t\right] = 0.$$

(c) [*10 marks*] A further jump condition is

$$[V_x\mathbf{B}_t - B_x\mathbf{V}_t] = 0.$$

(You need NOT prove this.)

Using these jump conditions show that the upstream and downstream transverse components of the magnetic field are parallel, i.e., that $\mathbf{B}_{dt} = \alpha\mathbf{B}_{ut}$ where α is a non-zero scalar. Hence, show that

$$\widehat{\mathbf{n}} \cdot (\mathbf{B}_u \times \mathbf{B}_d) = 0$$

i.e., that the three vectors $\widehat{\mathbf{n}}$, $\mathbf{B}_u$ and $\mathbf{B}_d$ are coplanar.

(d) [*3 marks*] From the previous result, and the further result (which you do not need to prove)

$$\widehat{\mathbf{n}} \cdot (\mathbf{B}_u - \mathbf{B}_d) = 0$$

explain why the vector

$$\mathbf{N} = (\mathbf{B}_u \times \mathbf{B}_d) \times (\mathbf{B}_u - \mathbf{B}_d)$$

can be used to determine the shock normal from the measured upstream and magnetic field vectors.