Lecture 10. Solution of the Friedman equation and the fate of the Universe

CONTENT

10.1. Asymptotic behavior of the solution of Friedman equation

Let us write down the Friedman equation in the following form (see Eq 4 of Lecture 9)

$$\dot{R}^2 = c^2 \left(\frac{R_*}{R} - k \right), \tag{1}$$

where

$$R_* = \frac{8\pi G \rho_0 R_0^3}{3c^2}. \tag{2}$$

To solve the Friedman equation let us use the method of separation of the variables:

$$\left(\frac{dR}{dt} \right)^2 = c^2 \left(\frac{R_*}{R} - k \right), \tag{3}$$

$$\frac{dR}{dt} = \pm c \sqrt{\frac{R_*}{R} - k}. \tag{4}$$

We should take "+" here because we deal with an expanding rather than a contracting Universe.

$$dt = + \frac{dR}{c\sqrt{\frac{R_*}{R} - k}}. \tag{5}$$

$$t = \frac{1}{c} \int_0^R \frac{dR'}{\sqrt{\frac{R_*}{R'} - k}}. \tag{6}$$

Let us consider the asymptotic behavior of the Friedman equation and its solution for small R, i.e in the beginning of the expansion of the Universe when

$$\frac{R_*}{R} \gg |k|, \quad \text{i.e.} \quad R \ll \frac{R_*}{|k|}. \tag{7}$$

If $k = 0$ this inequality is always valid. If $k \neq 0$ asymptotically the Friedman equation is the same as in the case $k = 0$.

This asymptotic solution can be easily found, indeed:

$$t = \frac{1}{c} \int_0^R \frac{dR'}{\sqrt{\frac{R_*}{R'}}} = \frac{1}{cR_*^{1/2}} \int_0^R R'^{1/2} dR' = \frac{2R^{3/2}}{3cR_*^{1/2}}, \tag{8}$$

hence the asymptotic solution goes as

$$R \sim t^{2/3}, \quad \text{if} \quad R \ll \frac{R_*}{|k|}. \tag{9}$$

10.2. Parametric solution of the Friedman equation with $k \neq 0$

The following relationships between between hyperbolic and trigonometric functions help a lot with solving the Friedman equation in the case when $k \neq 0$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{10}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{11}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \tag{12}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \tag{13}$$

hence

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = -\frac{e^{x} - e^{-x}}{2i} = -\frac{1}{i}\sinh x = i\sinh x \tag{14}$$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x \tag{15}$$

then

$$\cosh^2 x - \sinh^2 x = \cos^2(ix) - \left(\frac{\sin(ix)}{i}\right)^2 = \cos^2(ix) + \sin^2(ix) = 1. \tag{16}$$

Returning back to Eq. 6, we can use the following substitution:

$$R(\eta) = \frac{R_*}{k}\sin^2\frac{\sqrt{k}\eta}{2} = \frac{R_*}{2k}(1 - \cos\sqrt{k}\eta), \tag{17}$$

where η is a new variable. We should not worry that the argument $x = \sqrt{k}\eta$ could be a complex number because all sines and cosines can be expressed in terms of the hyperbolic functions of a real argument.
Then we have

$$t = \frac{R_*\sqrt{k}}{2kc}\int_0^{\eta}\frac{\sin\sqrt{k}\eta' d\eta'}{\sqrt{\frac{R_* k}{R_*\sin^2\frac{\sqrt{k}\eta'}{2}} - k}} = \frac{R_*}{2kc}\int_0^{\eta} d\eta' F(\sqrt{k}\eta'), \tag{18}$$

where

$$F(x) = \frac{\sin x}{\sqrt{\frac{1}{\sin^2\frac{x}{2}} - 1}} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}\sin\frac{x}{2}}{\sqrt{1 - \sin^2\frac{x}{2}}} = 2\sin^2\frac{x}{2} = 1 - \cos x. \tag{19}$$

Hence

$$t = \frac{R_*}{2kc}\int_0^\eta d\eta'(1-\cos\sqrt{k}\eta') = \frac{R_*}{2kc}(\eta - \frac{\sin\sqrt{k}\eta}{\sqrt{k}}). \tag{20}$$

Finally we have obtained the solution of Friedman equation in the following parametric form:

$$R = \frac{R_*}{2k}(1-\cos\sqrt{k}\eta), \tag{21}$$

$$t = \frac{R_*}{2kc}(\eta - \frac{\sin\sqrt{k}\eta}{\sqrt{k}}). \tag{22}$$

10.3. Three types of Newtonian cosmological models and the fate of the Universe

Thus according to Newtonian theory, there are three types of cosmological models:

$$\begin{array}{lcc} 1) & k=0 & \Omega_0 = 1, \\ 2) & k=-1 & \Omega_0 < 1, \\ 3) & k=1 & \Omega_0 > 1,. \end{array} \tag{23}$$

Let us consider all these models separately.

(i) $k = 0$.

In this case the asymptotic solution obtained in section 10.1 is always valid and we have explicit solution

$$t = \frac{2R^{3/2}}{3cR_*^{1/2}}, \quad R = R_*\left(\frac{t}{t_*}\right)^{2/3}, \tag{24}$$

where

$$t_* = \frac{2R_*}{3c}. \tag{25}$$

From the mass conservation equation (see the previous lecture) using Eqs. (2) and (24), we obtain the following expression for ρ:

$$\rho = \rho_0\left(\frac{R_0}{R}\right)^3 = \rho_0\left(\frac{R_0}{R_*}\right)^3\left(\frac{t}{t_*}\right)^{-2} = \frac{4}{9c^2}\rho_0 R_0^3(R_*)^{-1}t^2 = \frac{1}{6\pi G t^2}. \tag{26}$$

A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 4. PART II. Newtonian Cosmological Models. Lecture 10. Solutions of the Friedman equation and the fate of the Universe. 10.3. Three types of Newtonian cosmological models and the fate of the Universe.

(ii) $k = -1$.

Let us now consider the model with $k = -1$. From the parametric solution obtained in the previous section we have

$$R = \frac{R_*}{-2}(1 - \cos i\eta), \tag{27}$$

$$t = \frac{R_*}{-2c}(\eta - \frac{\sin i\eta}{i}, \tag{28}$$

From Eqs (14) and (15) we obtain

$$R = \frac{R_*}{2}(\cosh \eta - 1), \tag{29}$$

$$t = \frac{R_*}{2c}(\sinh \eta - \eta). \tag{30}$$

If $R \to \infty$ the parameter η also goes to infinity and

$$R \approx \frac{R_*}{2}\frac{e^\eta}{2}, \tag{31}$$

$$t \approx \frac{R_*}{2c}\frac{e^\eta}{2}, \tag{32}$$

Thus if $R \gg R_*$ we have another explicit asymptotic solution

$$R = ct, \tag{33}$$

which corresponds to the asymptotically free expansion of the Universe, i.e. with an asymptotically vanishing deceleration. In other words the model with $k = -1$ expands in the future for ever, asymptotically approaching the so called Milne solution describing the expansion of an empty Universe.

(iii) $k = 1$.

In the case $k = 1$ the performance of the solution is drastically different:

$$R = \frac{R_*}{2}(1 - \cos \eta), \tag{34}$$

$$t = \frac{R_*}{2c}(\eta - \sin \eta). \tag{35}$$

We can see that R attains maximum at $\eta = \pi$ and then the Universe will start to contract until $R = 0$. This is called the Big Crunch.
The behavior of all three cosmological models predicted by the Newtonian theory are summarized in (Fig. 10.1.