Lecture 12. Examples, problems and summary for Part II.

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12.1. Example for Lecture 7. Inflation

Given that the Hubble constant, H, does not depend on time. Find the scale factor $R(t)$ as a function of time. Show that in this case the Universe expands with an acceleration.

Solution

From

$$\frac{\dot{R}}{R} = H, \quad \dot{R} = HR \tag{1}$$

we have

$$R = R_0 e^{H(t-t_0)}. \tag{2}$$

and

$$\ddot{R} = H\dot{R} = H^2 R > 0. \tag{3}$$

12.2. Example for Lecture 8. Conservation of mass

Under what condition mass is conserved.

Solution

When pressure is negligible. Otherwise the forces of pressure produce a work, which change the energy within given volume. Energy E is related to mass M as

$$E = Mc^2, \tag{4}$$

Hence, the mass varies proportionally to the energy.

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12.3. Example for Lecture 9. The Hubble constant as a function of the scale factor

Use the Friedman equation for the Universe with $k = 0$ to find the Hubble constant H as a function of scale factor R if the present Universe contains only dust.
Solution

Taking into account that

$$H = \frac{\dot{R}}{R}, \quad \rho_{cr} = \frac{3H_0^2}{8\pi G}, \tag{5}$$

and

$$\rho = \frac{3H_0^2}{8\pi G}\left(\frac{R_0}{R}\right)^3, \tag{6}$$

the Friedman equation can be written as

$$H^2 = \frac{8\pi G}{3} \cdot \frac{3H_0^2}{8\pi G}\left(\frac{R_0}{R}\right)^3, \tag{7}$$

hence

$$H^2 = H_0^2\left(\frac{R_0}{R}\right)^3. \tag{8}$$

12.4. Example for Lecture 10. The parametric solution of the Friedman equation

Assume that the Universe is closed ($k = 1$) and contains only dust. Using the Friedman and energy conservation equations, verify that the evolution of the scale factor has the following parametric form:

$$R(\eta) = \frac{\beta}{2}(1 - \cos\eta), \quad t(\eta) = \frac{\beta}{2c}(\eta - \sin\eta).$$

Show that

$$\beta = \frac{2cq_0}{H_0(2q_0 - 1)^{\frac{3}{2}}},$$

where q_0 is the present deceleration parameter.

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Solution

The energy conservation equation in this case gives

$$\rho = \rho_0 (R_0/R)^3. \tag{9}$$

Substituting this result to the Friedman equation we have

$$\dot{R}^2 = c^2 \left(\gamma/R + 1\right), \tag{10}$$

where

$$\gamma = 8\pi\rho_0 R_0^3/3c^2. \tag{11}$$

Then we calculate $\dot{R}^2$, using the parametric solution:

$$\dot{R}^2 = \left[\frac{d\left[\frac{\beta}{2}(\cosh\eta - 1)\right]}{d\left[\frac{\beta}{2c}(\sinh\eta - \eta)\right]}\right]^2 = c^2 \frac{\sinh^2\eta}{(\cosh\eta - 1)^2} = c^2 \frac{1+\cosh\eta}{\cosh\eta - 1}. \tag{12}$$

Putting this into the Friedman equation we have

$$c^2 \frac{1+\cosh\eta}{\cosh\eta - 1} = c^2 \left(\frac{2\gamma}{\beta(\cosh\eta - 1)} + 1\right), \quad 1 + \cosh\eta = \frac{2\gamma}{\beta} + 1 + \cosh\eta. \tag{13}$$

We see that this parametric solution does satisfies the Friedman equation, if

$$\beta = \gamma = 8\pi\rho_0 R_0^3/3c^2. \tag{14}$$

From the Friedman equation, taken at the moment t_0 we have

$$H_0^2 R_0^2 = H_0^2 \Omega_0 + c^2, \tag{15}$$

hence we can express R_0 in terms of H_0 and Ω_0 as

$$R_0 = \frac{c}{H_0\sqrt{1-\Omega_0}}. \tag{16}$$

Then substituting this to the formula for β we have

$$\beta = \frac{c\Omega_0}{H_0(1-\Omega_0)^{3/2}}. \tag{17}$$

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12.5. Example for Lecture 11. Gravitational paradox

To illustrate that Newtonian theory is not applicable to the infinite Universe, show that gravitational forces from two halves of the homogeneous Universe are infinite and should destroy any body (a contradiction with an every day experience).

Solution

Dividing the Universe into infinite planes

$$|| \qquad\qquad \longleftarrow \bigcirc \longrightarrow \quad z \qquad\qquad ||$$

in cylindrical coordinates we have

$$dF_z(z) = Gm\rho dz \int_0^{2\pi} d\phi \int_0^{\infty} \frac{rdr\cos\theta}{R^2}. \tag{18}$$

Taking into account that

$$R = \sqrt{r^2 + z^2}, \quad \cos\theta = \frac{z}{R}, \tag{19}$$

we obtain

$$dF_z(z) = 2\pi Gm\rho z dz \int_0^{\infty} \frac{rdr}{(r^2+z^2)^{3/2}} = 2\pi Gm\rho dz. \tag{20}$$

Hence,

$$F_z = \int_0^{\infty} dF(z) = 2\pi Gm\rho\infty_0 dz = \infty. \tag{21}$$

12.6. Summary of Part II.

(i) We obtained reasonable cosmological models predicting the future and explaining some global properties of the Universe in the past.

(ii) But we need GR to decide what is wright and what is wrong in predictions made by Newtonian cosmological models.