A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity. Lecture 13. The Principle of Equivalence and Geometrical Principle.

Lecture 13. The Principle of Equivalence and Geometrical Principle

CONTENT

13.1. The Principle of Equivalence

The basic postulate of the General Relativity states that a uniform gravitational field is equivalent to (which means is not distinguishable from) a uniform acceleration. A person cannot see locally the difference between standing on the surface of some gravitating body (for example the Earth) and moving in a rocket with corresponding acceleration (see Fig.13.1.

All bodies in given gravitational field move in the same manner, if initial conditions are the same. In other words, in given gravitational field all bodies move with the same acceleration. In absence of gravitational field, all bodies move also with the same acceleration relative to the non-inertial frame. Thus we can formulate the Principle of Equivalence which says: locally, any non-inertial frame of reference is equivalent to a certain gravitational field.

The important consequence of the Principle of Equivalence is that locally gravitational field can be eliminated by proper choice of the frame of reference. Such frames of reference are called locally inertial or galilean frames of reference. There is no experiment to distinguish between being weightless far out from gravitating bodies in space and being in free-fall in a gravitational field (see Fig.13.2.

Globally (not locally), "actual" Gravitational Fields can be distinguished from corresponding non-inertial frame of reference by its behavior at infinity: Gravitational Fields generated by gravitating bodies fall with distance.

13.2. The Principle of Equivalence in Newtonian Theory

In Newton's theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi, \tag{1}$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass, which is the gravitational analog of the electric charge in the theory of electromagnetism. The fundamental property of gravitational fields that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1, \tag{2}$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

13.3. The Principle of Equivalence in GR

Special Relativity is the theory which is valid only if we work within very special frames of reference (the global inertial frames). For such frames of reference the following combination of time and space coordinates remains invariant whatever global inertial frame of references is chosen

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{3}$$

This combination is called the interval. All space-time coordinates in different global inertial frames of reference are related with each other by the Lorentz transformations which leave the shape of the interval unchanged. But this is not the case if one considers transformation of coordinates in more general case, when at least one of the two frames of reference is non-inertial. This interval is not reduced anymore to the simple sum of squares of the coordinate differentials and can be written in the following more general quadratic form:

$$ds^2 = g_{ik}dx^i dx^k \equiv \sum_{i=0}^{3}\sum_{k=0}^{3} g_{ik}dx^i dx^k, \tag{4}$$

where repeating indices mean summation. In inertial frames of reference

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \text{ and } g_{ik} = 0, \quad \text{if} \quad i \neq k. \tag{5}$$

Example. Transformation to uniformly rotating frame.

$$x = x' \cos \Omega t - y' \sin \Omega t, \;\; y = x' \sin \Omega t + y' \cos \Omega t, \;\; z = z', \tag{6}$$

where Ω is the angular velocity of rotation around z-axis. In this non-inertial frame of reference

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt. \tag{7}$$

13.4. Gravity as a space-time geometry

The fundamental physical concept of General Relativity is that gravitational field is identical to geometry of curved space-time. This idea, called the Geometrical principle, entirely determines the mathematical structure of General Relativity.
According to GR gravity is nothing but manifestation of space-time 4-geometry. The geometry is determined by the interval

$$ds^2 = g_{ik}(x^m)dx^i dx^k, \tag{8}$$

where $g_{ik}(x^m)$ is called the metric tensor. What is exactly meant by the term "tensor" we will discuss later. At the present moment we can consider $g_{ik}(x^m)$ as a 4×4 - matrix and all its components in general case can depend on all 4 coordinates x^m, where $m = 0, 1, 2, 3$. All information about the geometry of space-time is contained in $g_{ik}(x^m)$.The dependence of $g_{ik}(x^m)$ on x^m means that this geometry is different in different events, which implies that the space-time is curved and its geometry is not Euclidian. Such sort of geometry is the the subject of mathematical discipline called Differential Geometry developed in XIX Century.

The GR gives very simple and natural explanation of the Principle of Equivalence: In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in flat space-time when free particles move along straight lines which are geodesics in flat space-time.
The one of the main statements of General Theory of Relativity is the following: If we know g_{ik}, we can determine completely the motion of test particles and performance of all test fields. [Test particle or test field means that gravitational field generated by these test objects is negligible.] The metric tensor g_{ik} itself is determined by physical content of the space-time.
In any curved space-time (i.e in the actual gravitational field) there is no global galilean frames of reference. In flat space-time, if me work in non-inertial frames of reference metrics looks like the metric in gravitational field (because according to the Equivalence Principle locally actual gravitational field is not distinguishable from corresponding non-inertial frame of reference), nevertheless local galilean frames of reference do exist. The local galilean frame of reference is equivalent to the freely falling frame of reference in which locally gravitational field is eliminated. From geometrical point of view to eliminate gravitational field locally means to find such frame of reference in which

$$g_{ik} \to \eta_{ik} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{9}$$

Thus, according to General Relativity gravitational field is identical to geometry of curved space-time (see Fig.13.3.and Fig.13.4. for examples of highly curved space-time. These pictures are also taken from the very interesting astronomical web-site of Nick Strobel.
This fundamental physical concept of General Relativity, identity of the gravitational field and the geometry of curved space-time is called the Geometrical principle.This principle entirely predetermines the mathematical structure of General Relativity.

13.5. Geodesics

If space-time is flat and one works with inertial frames of reference the world lines of free particles are straight lines (another formulation of the first law of Newton). For particles moving with acceleration the world lines are curved (seeFig.13.5.
The fact that all bodies move with the same acceleration in given gravitational field seems to be now absolutely clear, because all bodies in given gravitational field moves along the same geodesics (the world lines which are the most close to the straight world lines). The shape of geodesics is determined only by geometry of space-time itself.