A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity.Lecture 14. Physical Geometry of Space-Time.

Lecture 14. Physical Geometry of Space-Time.

CONTENT

A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity.Lecture 14. Physical Geometry of Space-Time.14.1. Proper time. 14.2. Spatial distance.

One of the most central problems in the geometry of 4-spacetime can be formulated as follows. If the metric tensor is given, how actual (measurable) time and distances are related with coordinates x^0, x^1, x^2, x^3 chosen in arbitrary way.

14.1. Proper time

Let us consider the world line of an observer who uses some clock to measure the actual or proper time $d\tau$ between two infinitesimally close events in the same place in space. Obviously we should put

$$dx^1 = dx^2 = dx^3 = 0. \tag{1}$$

Let us define proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c}, \tag{2}$$

then we have

$$ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00}(dx^0)^2, \tag{3}$$

thus

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0. \tag{4}$$

For the proper time between any two events (not necessary that these events are infinitesimally close) occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0. \tag{5}$$

14.2. Spatial distance

Separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2 = C + 2By + Ay^2, \tag{6}$$

where

$$C = g_{\alpha\beta}dx^\alpha dx^\beta, \quad B = g_{0\alpha}dx^\alpha, \quad A = g_{00} \text{ and } y = dx^0. \tag{7}$$

To define dl we will use a light signal according to the following procedure: From some point B with spatial coordinates $x^\alpha + dx^\alpha$ a light signal emitted at the moment corresponding to time coordinate $x^0 + dx^{0(1)}$ propagates to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + dx^{0(2)}$ as shown below.

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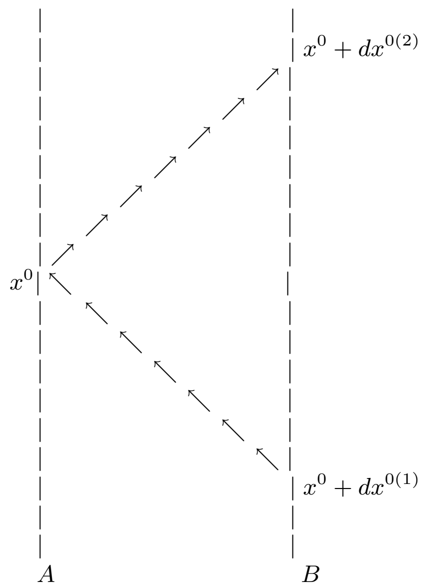

The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero:

$$ds = 0, \quad \text{i.e.} \quad C + 2By + Ay^2 = 0. \tag{8}$$

Solving this quadratic equation with respect to $y = dx^0$ we find two roots:

$$dx^{0(1)} = \frac{-B - \sqrt{B^2 - AC}}{A} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right) \tag{9}$$

$$dx^{0(2)} = \frac{-B + \sqrt{B^2 - AC}}{A} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right) \tag{10}$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}. \tag{11}$$

Then

$$dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)}) \tag{12}$$

and finally

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \quad \text{where} \quad \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}. \tag{13}$$

Thus, if we know g_{ik}, which is the pure geometrical object, we can determine proper time and physical spatial distance, using physical procedure of sending light signal. This is a really good example of relationship between Geometry and Physics.