Lecture 15. The Principle of Covariance and Tensors.

CONTENT

	Page
15.1. The Principle of Covariance	60
15.2. Summation convention	60
15.3. Transformation of coordinates	62
15.4. Vectors	63
15.5. Tensors	64
15.6. Reciprocal tensors	65

A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity.Lecture 15. The Principle of Covariance and Tensors.15.1. The Principle of Covariance. 15.2. Summation convention.

15.1. The Principle of Covariance

This Principle of Covariance says:

The shape of all physical equations should be the same in an arbitrary frame of reference.

Otherwise the physical equations being different in gravitational field and in inertial frames of reference would have different solutions. In other words, these equations would predict the difference between gravitational field and non-inertial frame of reference and hence, would contradict to experimental data. This principle refers to the most general case of non-inertial frames (in contrast to the Special Theory of Relativity which works only in inertial frames of reference). Hence this principle is nothing but more mathematical formulation of the Principle of Equivalence that there is no way experimentally to discriminate between gravitational field and non-inertial frame of reference. This principle set very severe requirements to all physical equations and predetermines the mathematical structure of General Relativity: all equations should contain only tensors. By definition, tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another. Taking into account that non-inertial frames of reference in 4-dimensional space-time correspond to curvilinear coordinates, it is necessary to develop four-dimensional differential geometry in arbitrary curvilinear coordinates.

15.2. Summation convention

The Einstein notation (introduced in 1916) or Einstein summation convention is a notational convention useful when dealing with long coordinate formulae of General relativity.

According to this convention, when an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values. In our course, which is about the four dimensional space-time, the indices are 0,1,2,3 (0 represents the time coordinate). We will use the Roman alphabet for such indices. For spatial indices, 1,2,3, we will use the Greek alphabet. (In other textbooks Roman and Greek may be reversed.)

A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity.Lecture 15. The Principle of Covariance and Tensors.15.2. Summation convention.

Example 1.

$$A^i B_i \equiv \sum_{i=0}^{3} A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3, \tag{1}$$

$$A^\alpha B_\alpha \equiv \sum_{i=1}^{3} A^\alpha B_\alpha = A^1 B_1 + A^2 B_2 + A^3 B_3. \tag{2}$$

One must distinguish between superscripts and subscripts.

Example 2.

The following expressions don't imply summation:

$$A_i B_i, \;\; A^i B^i, \;\; A_\alpha B_\alpha \;\; A^\alpha B^\alpha. \tag{3}$$

Free indices are indices which don't participate in summation, i.e. free indices can not appear twice in the same term. Notation and position of free indices in left and right side of equations should be the same.

Example 3.

The following expressions are written properly:

$$A_i = B_i, \;\; A^i = B^i, \;\; A_\alpha = B_\alpha \;\; A^\alpha = B^\alpha. \tag{4}$$

The following expressions are wrong and have no sense:

$$A_i = B^i, \;\; A^i = B_i, \;\; A_n = B_i \;\; A^\alpha = B^\beta. \tag{5}$$

Repeating indices should appear twice as explained before and can not appear in the same term more than twice.

Example 4.

The following expressions are written properly:

$$A^i B_i C_m P^m, \;\; A^i B^m C_m. \tag{6}$$

The following expressions are wrong and have no sense:

$$A^i B_i C_i P^m, \;\; A^i B_i C_i P^i, . \tag{7}$$

$$A_i = B^i, \;\; A^i = B_i, \;\; A_n = B_i \;\; A^\alpha = B^\beta. \tag{8}$$

Summation convention saves time, chalk and helps very much if you manipulate properly.

Example 5:

The following equation

$$A^i B_k C_m = P^v L_{vnj} G^{nj} Q^i_{km}, \tag{9}$$

corresponds to 64 equations with 64 terms on right hand side of each equation!!!

15.3. Transformation of coordinates

Let us consider the transformation of coordinates from one frame of reference (x^0, x^1, x^2, x^3) **to another,**($x^{'0}$, $x^{'1}$, $x^{'2}$, $x^{'3}$):

$$x^0 = f^0(x'^0, x'^1, x'^2, x'^3), \tag{10}$$

$$x^1 = f^1(x'^0, x'^1, x'^2, x'^3), \tag{11}$$

$$x^2 = f^2(x'^0, x'^1, x'^2, x'^3), \tag{12}$$

$$x^3 = f^1(x'^0, x'^1, x'^2, x'^3). \tag{13}$$

Then

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k = S^i_k dx'^k, \;\; i, k = 0,\ 1,\ 2,\ 3, \tag{14}$$

where

$$S^i_k = \frac{\partial x^i}{\partial x'^k} \tag{15}$$

is a transformation matrix. Remember that all repeating indices mean summation, otherwise even such basic transformation would be written ugly. To demonstrate that summation convention is really very useful, I will write, the first and the last time, the same transformation without using the summation convention

$$dx^0 = \frac{\partial x^0}{\partial x'^0}dx'^0 + \frac{\partial x^0}{\partial x'^1}dx'^1 + \frac{\partial x^0}{\partial x'^2}dx'^2 + \frac{\partial x^0}{\partial x'^3}dx'^3 = \tag{16}$$

$$= S^0_0 dx'^0 + S^0_1 dx'^1 + S^0_2 dx'^2 + S^0_3 dx'^3, \tag{17}$$

$$dx^1 = \frac{\partial x^1}{\partial x'^0}dx'^0 + \frac{\partial x^1}{\partial x'^1}dx'^1 + \frac{\partial x^1}{\partial x'^2}dx'^2 + \frac{\partial x^1}{\partial x'^3}dx'^3 = \tag{18}$$

$$= S^1_0 dx'^0 + S^1_1 dx'^1 + S^1_2 dx'^2 + S^1_3 dx'^3, \tag{19}$$

$$dx^2 = \frac{\partial x^2}{\partial x'^0}dx'^0 + \frac{\partial x^2}{\partial x'^1}dx'^1 + \frac{\partial x^2}{\partial x'^2}dx'^2 + \frac{\partial x^2}{\partial x'^3}dx'^3 = \tag{20}$$

$$= S^2_0 dx'^0 + S^2_1 dx'^1 + S^2_2 dx'^2 + S^2_3 dx'^3, \tag{21}$$

$$dx^3 = \frac{\partial x^3}{\partial x'^0}dx'^0 + \frac{\partial x^3}{\partial x'^1}dx'^1 + \frac{\partial x^3}{\partial x'^2}dx'^2 + \frac{\partial x^3}{\partial x'^3}dx'^3 = \tag{22}$$

$$= S^3_0 dx'^0 + S^3_1 dx'^1 + S^3_2 dx'^2 + S^3_3 dx'^3. \tag{23}$$

15.4. Vectors

Now we can give the definition of the Contravariant four-vector:

The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:

$$A^i = S^i_k A'^k. \tag{24}$$

Let φ is scalar field, then

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i} = \tilde{S}^k_i \frac{\partial \varphi}{\partial x'^k}, \tag{25}$$

where $\tilde{S}^k_i$ is another transformation matric. What is relation of this matrix with the previous transformation matrix S^i_k? If we take product of these matrices, we obtain

$$S^i_n \tilde{S}^n_k = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^n}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k, \tag{26}$$

where δ^i_k is so called Kronneker symbol, which actually is nothing but the unit matrix:

$$\delta^i_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

In other words $\tilde{S}^i_k$ is inverse or reciprocal with respect to S^i_k.

Now we can give the definition of the Covariant four-vector:
The Covariant four-vector is the combination of four quantities (components) A_i, which are transformed like like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \tag{28}$$

Note, that for contravariant vectors we always use upper indices, which are called contravariant indices, while for covariant vectors we use low indices, which are called covariant indices. In General Relativity summation convention always means that on of the two repeating indices should be contravariant and another should be covariant. For example,

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \ \ \textbf{(scalar product)}, \tag{29}$$

while there is no summation if both indices are, say, covariant:

$$A_i B_i = \begin{array}{ll} A_0 B_0, & \text{if } i = 0 \\ A_1 B_1, & \text{if } i = 1 \\ A_2 B_2, & \text{if } i = 2 \\ A_3 B_3, & \text{if } i = 3 \end{array} \tag{30}$$

15.5. Tensors

Now we can generalize the definitions of vectors and introduce tensors entirely in terms of transformation laws.

Scalar is the tensor of the 0 rank. It has only $4^0 = 1$ component and 0 number of indices. Transformation law is

$$A = A', \tag{31}$$

we see that transformation matrices appear in transformation law 0times.

A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity.Lecture 15. The Principle of Covariance and Tensors.15.5. Tensors. 15.6. Reciprocal tensors.

Contravariant and covariant vectors are tensors of the 1 rank. They have $4^1 = 4$ components and 1 index. Corresponding transformation laws are

$$A^i = S^i_n A'^n, \tag{32}$$

$$A_i = \tilde{S}^n_i A'_n, \tag{33}$$

we see only 1 transformation matrix in each transformation law.

Contravariant tensor of the 2 rank has $4^2 = 16$ components and 2 contravariant indices. Corresponding transformation law is

$$A^{ik} = S^i_n S^k_m A'^{nm}, \tag{34}$$

we see 2 transformation matrices in the transformation law.

Covariant tensor of the 2 rank has $4^2 = 16$ components and 2 covariant indices. Corresponding transformation law is

$$A_{ik} = \tilde{S}^n_i \tilde{S}^m_k A'_{nm}, \tag{35}$$

we see 2 transformation matrices in the transformation law.

Mixed tensor of the 2 rank has $4^2 = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m, \tag{36}$$

we see 2 transformation matrices in the transformation law.

Covariant tensor of the 3 rank has $4^3 = 64$ components and 3 covariant indices. Corresponding transformation law is...

Mixed tensor of the $N + M$ rank with N contravariant and M covariant indices, has $4^{N+M} = 2^{2(N+M)}$ components and $N + M$ indices. Corresponding transformation law is

$$A^{i_1\, i_2\, \dots\, i_N}_{k_1\, k_2\, \dots\, k_M} = S^{i_1}_{n_1} S^{i_2}_{n_2} \dots S^{i_N}_{n_N} \tilde{S}^{m_1}_{k_1} \tilde{S}^{m_2}_{k_2} \dots \tilde{S}^{m_M}_{k_M} A'^{n_1\, n_2\, \dots\, n_N}_{m_1 m_2 \dots m_M}, \tag{37}$$

we see N+M transformation matrices in the transformation law.

15.6. Reciprocal tensors

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik} B^{kl} = \delta^l_i. \tag{38}$$

A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 5. PART III. Mathematical structure of General Relativity.Lecture 15. The Principle of Covariance and Tensors.15.6. Reciprocal tensors.

We can introduce now a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta^l_i. \tag{39}$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k, \tag{40}$$

in other words we can rise and descend indices as we like, some sort of juggling with indices. We can say that contravariant, covariant and mixed tensors can be considered as different representations of the same geometrical object.

For the contravariant metric tensor itself we have very important representation in terms of the transformation matrix from locally inertial frame of reference (galilean frame) to an arbitrary non-inertial frame, let us denote it as $S^i_{(0)k}$. We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1), \tag{41}$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{nm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}. \tag{42}$$

This means that if we know the transformation law from the local galilean frame of reference to an arbitrary frame of reference, we know the metric at this arbitrary frame of reference and, hence, gravitational field!