Chapter 18. Examples, problems and summary for Part III.

CONTENT

18.1. Examples on the transformations of tensors

a) Give the definition of the mixed tensor of the second rank, A^i_k, and b) the mixed tensor of the third rank, B^i_{km}. c) Given that in the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the fourth rank, C^{ik}_{lm} has the only one non-vanishing component, $C^{00}_{00[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference, x^i, in terms of the transformation matrices $S^l_{m[G]} = \frac{\partial x^l}{\partial x^m_{[G]}}$ and $\tilde{S}^l_{m[G]} = \frac{\partial x^l_{[G]}}{\partial x^m}$.

Solutions

a) The mixed tensor of the second rank is the object containing $4^2 = 16$ components A_{ik} which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

$$A^i_k = S^i_v \tilde{S}^u_k A'_{vu},$$

where

$$\tilde{S}^l_m = \frac{\partial x'^l}{\partial x^m}.$$

b) The mixed tensor of the third rank with one contravariant and two covariant indices is the object containing $4^3 = 64$ components B^i_{km} which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

c) The law of transformation for the tensor C^{ik}_{lm} from local Galilean to arbitrary frame of reference is

$$C^{ik}_{lm} = S^i_p S^k_v \tilde{S}^u_l \tilde{S}^w_m C^{pv}_{uw(G)},$$

where

$$S^l_m = S^l_{(G)m} = \frac{\partial x^l}{\partial x^m_G}$$

As given

$$C^{pv}_{uw(G)} = \delta^p_0 \delta^v_0 \delta^1_u \delta^1_w,$$

hence

$$C^{ik}_{lm} = S^i_0 S^k_0 \tilde{S}^1_l \tilde{S}^1_m.$$

18.2. Examples on the Ricci and the Stress-Energy tensors.

Using the EFEs and Bianchi identity (see rubric) show that the stress-energy tensor satisfies conservation law $T^i_{k:i} = 0$.

Solutions

After contracting the Biancci identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0.$$

According to the definition of Ricci tensor

$$R^i_{kil} = R_{kl},$$

the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}.$$

Taking into account that the Riemann tensor is antisymmetric with respect permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km},$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}.$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip} R_{pklm;i},$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip} R_{pklm;i} = -g^{ip} R_{kplm;i} = g^{ip} R_{kpml;i}.$$

After all these manipulations we have

$$g^{ip} R_{kpml;i} + R_{kl;m} - R_{km;l} = 0.$$

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km} g^{ip} R_{kpml;i} + g^{km} R_{kl;m} - g^{km} R_{km;l} = \left(g^{km} g^{ip} R_{kpml}\right)_{;i} + \left(g^{km} R_{kl}\right)_{;m} - \left(g^{km} R_{km}\right)_{;l} = 0.$$

In the first term expression in brackets can be simplified as

$$g^{km} g^{ip} R_{kpml} = g^{ip} R_{pl} = R^i_l.$$

In the second term expression in brackets can be simplified as

$$g^{km} R_{kl} = R^m_l.$$

According to the definition of scalar curvature

$$R = g^{km} R_{km},$$

the third term can be simplified as

$$\left(g^{km} R_{km}\right)_{;l} = R_{;l} = R_{,l}.$$

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0,$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0, \text{ or } R^i_{l;i} - \frac{1}{2} R_{,l} = 0.$$

Using the EFEs in the mixed form

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

we obtain

$$T^i_{k;i} = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right)_{;i} = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2}\delta^i_k R_{,i}\right) = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2} R_{,k}\right) = 0.$$

18.3. Examples on the scalar curvature and the trace of Stress-Energy Tensor

a) Show that the covariant divergence of the stress-energy tensor is equal to zero, $T^i_{k\,;\,k} = 0$. b) The stress-energy tensor has the following form

$$T_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix},$$

where ε is energy density and P is pressure (if $P > 0$) or tension (if $P < 0$). Using the Einstein equations express the scalar curvature in terms of ε and P.

Solutions

a) Multiplying the EFEs

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik}$$

by g^{mk} we obtain

$$R^m_i - \frac{1}{2}\delta^m_i R = \frac{8\pi G}{c^4} T^m_k.$$

Taking covariant divergence of LHS and RHS of this equation we obtain

$$R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m} = \frac{8\pi G}{c^4} T^m_{k;\,m},$$

hence

$$T^m_{k;\,m} = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m}\right) = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2} R_{,i}\right) = 0.$$

b) Contracting the EFEs written in mixed form (we have

$$R^m_m - \frac{1}{2}\delta^m_m R = \frac{8\pi G}{c^4} T^m_m,$$

hence

$$R - \frac{4}{2}R = \frac{8\pi G}{c^4} T = \frac{8\pi G}{c^4}(\varepsilon - 3P),$$

hence

$$R - 2R = \frac{8\pi G}{c^4}(\varepsilon - 3P),$$

finally

$$R = -\frac{8\pi G}{c^4}(\varepsilon - 3P).$$

18.4. Summary of Part III

1) The Equivalence principle leads to the Principle of Covariance

2) Gravity is manifestation of the geometry of space-time (Geometrical principle)

3) These principles predetermine the whole mathematical structure of GR.

4) To take into account any effect of gravity on any physical process it is enough to make replacement

$$d \to D, \quad , \to ;$$

5) EFEs describe the generation of gravity by matter and we will use EFEs in Cosmology.