PART IV. Relativistic Cosmological Models.

Chapter 19. Derivation of Relativistic Cosmological Models.

CONTENT

19.1. Geometry of isotropic and homogeneous Universe

What is the space-time geometry corresponding to isotropic and homogeneous Universe? Let us write the metric in the following form:

$$ds^2 = g_{ik}dx^i dx^k = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0 dx^\alpha - dl^2, \quad (1)$$

where

$$dl^2 = -g_{\alpha\beta}dx^\alpha dx^\beta. \quad (2)$$

We will work in the so called "co-moving" frame of reference, co-moving to the matter filling the Universe. The fact that all directions in a isotropic Universe are equivalent to each other, means that

$$g_{0\alpha} = 0, \quad (3)$$

otherwise $g_{0\alpha} \neq 0$ considered as 3-vector would lead to non-equivalence of different directions. The homogeneity of the Universe means that g_{00} can depend only on time coordinate, hence we can choose time coordinate t as

$$c^2 dt^2 = g(x^0)_{00}(dx^0)^2, \quad (4)$$

to obtain

$$ds^2 = c^2 dt^2 - dl^2. \quad (5)$$

19.2. Three-dimensional space of constant curvature

According to the cosmological principle the Universe is the same everywhere, as a consequence, The three-dimensional space is curved in the same way everywhere, which means that at each moment of time the metric of the space is the same at all points. To obtain such a metric let us start from the following geometrical analogy. Let us consider the two-dimensional sphere in the flat three-dimensional space. In this case the element of length is

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \quad (6)$$

The equation of a sphere of radius a in the three-dimensional space has the form

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = a^2. \quad (7)$$

The element of length on the two-dimensional sphere can be obtained if one expresses dx^3 in terms of dx^1 and dx^2. From the equation for sphere we have

$$x^1dx^1 + x^2dx^2 + x^3dx^3 = 0, \tag{8}$$

hence

$$dx^3 = -\frac{x^1dx^1 + x^2dx^2}{x^3} = -\frac{x^1dx^1 + x^2dx^2}{\sqrt{a^2 - (x^1)^2 - (x^2)^2}}. \tag{9}$$

Substituting dx^3 into dl we have

$$dl^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1dx^1 + x^2dx^2)^2}{a^2 - (x^1)^2 - (x^2)^2}. \tag{10}$$

Introducing the "polar" coordinates instead of x^1 and x^2

$$x^1 = r\cos\phi,$$

$$x^2 = r\sin\phi, \tag{11}$$

we obtain

$$dl^2 = (dr\cos\phi - r\sin\phi d\phi)^2 + (dr\sin\phi + r\cos\phi d\phi)^2 +$$
$$\frac{r\cos\phi(dr\cos\phi - r\sin\phi d\phi) + r\sin\phi(dr\sin\phi + r\cos\phi d\phi)}{a^2 - r^2} = \frac{a^2dr^2}{a^2 - r^2} + r^2d\phi^2 =$$
$$= \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2d\phi^2. \tag{12}$$

Now we can repeat step by step the previous derivation, by considering the geometry of the three-dimensional space as the geometry on the three-dimensional hypersphere in some fictitious four-dimensional space (don't confuse with the physical four-dimensional space-time). In this case the element of length is

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \tag{13}$$

The equation of a sphere of radius a in the four-dimensional space has the form

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2. \tag{14}$$

The element of length on the three-dimensional hypersphere, which represents the three-dimensional space of constant curvature, can be obtained, if one expresses dx^4 in terms of dx^1, dx^2 and dx^3. From the equation for hypersphere we have

$$x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4 = 0, \tag{15}$$

hence

$$dx^4 = -\frac{x^1dx^1 + x^2dx^2 + x^3dx^3}{x^4} = -\frac{x^1dx^1 + x^2dx^2 + x^3dx^3}{\sqrt{a^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}}. \tag{16}$$

Substituting dx^3 into dl we have

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{(x^1dx^1 + x^2dx^2 + x^3dx^3)^2}{a^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}. \tag{17}$$

Introducing the "spherical" coordinates instead of x^1, x^2 and x^3

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi, \\ x^2 &= r\sin\theta\sin\phi, \\ x^3 &= r\cos\theta, \end{aligned} \tag{18}$$

we obtain

$$\begin{gathered} dl^2 = (dr\sin\theta\cos\phi + r\cos\theta d\theta - r\sin\theta\sin\phi d\phi)^2 + \\ +(dr\sin\theta\sin\phi + r\cos\theta\cos\phi d\theta + r\sin\theta\sin\phi d\phi)^2 + (dr\cos\theta - r\sin\theta d\theta)^2 + \\ \frac{1}{a^2 - r^2}[r\sin\theta\cos\phi(dr\sin\theta\cos\phi + r\cos\theta d\theta - r\sin\theta\sin\phi d\phi) + \\ +r\sin\theta\sin\phi(dr\sin\theta\sin\phi + r\cos\theta\cos\phi d\theta + r\sin\theta\sin\phi d\phi) + \\ +r\cos\theta(dr\cos\theta - r\sin\theta d\theta)] = \\ = \frac{a^2dr^2}{a^2 - r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = \\ = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{gathered} \tag{19}$$

Taking into account that $r \leq a$ we can introduce instead of r the new lagrangian radial coordinate χ such that

$$r = a\sin\chi \ \text{ and } \ dr = a\cos\chi d\chi, \tag{20}$$

as a result dl can be rewritten as

$$dl^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{21}$$

Now we can write the metric interval for the four dimensional space time as

$$ds^2 = c^2dt^2 - a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{22}$$

We can repeat all these calculation for the three-dimensional space of the negative constant curvature. For that one should replace the equation for the hypersphere by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -a^2, \tag{23}$$

which is a sphere of imaginary radius. This means that to obtain the metric of three-dimensional space of constant negative curvature one should just replace a by ia.
Obviously, when $a \to \infty$ we obtain the case of the spatially flat space.
This method is called the method of embedding diagrams. The Geometry on different surfaces of constant curvature is shown in Fig.19.1. We will see later then in the relativistic cosmology the curvature of the three-dimensional space is related with the density parameter as it is shown on this figure.

19.3. Friedman-Lematre-Robertson-Walker (FLRW) metric

The fact that the Universe is expanding means that instead of constant a we should introduce a scale factor $R(t)$ and finally we obtain the famous Friedmann-Lematre-Robertson-Walker metric for expanding Universe

$$ds^2 = c^2dt^2 - R^2(t)[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)], \tag{24}$$

where

$$f = \left\{ \begin{array}{cc} \sin\chi & \text{for constant positive curvatue} \\ \sinh\chi & \text{for constant negative curvatue} \\ \chi & \text{for zero curvature} \end{array} \right\}. \tag{25}$$

The function $f(\chi)$ can be written in more elegant way as follows

$$f(\chi) = \frac{\sin A\chi}{A}, \tag{26}$$

where

$$A = \left\{ \begin{array}{cc} 1 & \text{for constant positive curvature} \\ i & \text{for constant negative curvature} \\ 0 & \text{for zero curvature} \end{array} \right\}. \tag{27}$$

Indeed, if $A = 1$, obviously $f(\chi) = \sin\chi$. If $A = i$

$$f(\chi) = \frac{\sin i\chi}{i} = \frac{e^{i\cdot iA} - e^{-i\cdot iA}}{2i\cdot i} = \frac{e^{-A} - e^{A}}{-2} = \frac{e^{A} - e^{-A}}{2} = \sinh\chi. \tag{28}$$

If $A = 0$

$$f(\chi) = \lim_{A\to 0} \frac{\sin A\chi}{A} = \chi. \tag{29}$$

Sometime it is convenient to introduce another time coordinate, η, called the conformal time and defined as

$$cdt = Rd\eta. \tag{30}$$

Then

$$ds^2 = R(\eta)[d\eta^2 - d\chi^2 - f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{31}$$

Now using the EFEs we can obtain required equations for $R(t)$, in other words we can obtain relativistic cosmological models based on the EFEs.

19.4. Relativistic acceleration equation

In order to avoid confusion between the scale factor R used in previous sections, with the scale curvature used in the present section, let us use a as the new notation for the scale factor. Contracting the EFEs written in mixed form we have

$$R^m_m - \frac{1}{2}\delta^m_m R = \frac{8\pi G}{c^4}T^m_m, \tag{32}$$

hence

$$R - \frac{4}{2}R = \frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(\varepsilon - 3P), \tag{33}$$

hence

$$R - 2R = \frac{8\pi G}{c^4}(\varepsilon - 3P), \tag{34}$$

finally

$$R = -\frac{8\pi G}{c^4}(\varepsilon - 3P). \tag{35}$$

Then

$$R^i_k = \frac{1}{2}\delta^i_k R + \frac{8\pi G}{c^4}T^i_k = \frac{8\pi G}{c^4}T^i_k - \frac{1}{2}\delta^i_k\frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(T^i_k - \frac{1}{2}\delta^i_k T). \tag{36}$$

$$R^0_0 = \frac{8\pi G}{c^4}(T^0_0 - \frac{1}{2}T) = \frac{8\pi G}{c^4}[\varepsilon - \frac{1}{2}(\varepsilon - 3P)] = \frac{4\pi G}{c^4}(\varepsilon + 3P) = \frac{4\pi G}{c^2}(\rho + \frac{3P}{c^2}). \tag{37}$$

$$R^0_0 = g^{0n}R_{n0} = R_{00} = \Gamma^l_{00,l} - \Gamma^l_{0l,0} + \Gamma^l_{00}\Gamma^m_{lm} - \Gamma^m_{0l}\Gamma^l_{0m}. \tag{38}$$

We can see that

$$\Gamma^l_{00} = \frac{g^{ln}}{2}\left(g_{0n,0} + g_{n0,0} - g_{00,n}\right) = g^{ln}\left(g_{0n,0} - \frac{1}{2}g_{00,n}\right) = g^{ln}g_{0n,0} = g^{l0}g_{00,0} + g^{l\alpha}g_{0\alpha,0} = 0. \tag{39}$$

Hence

$$R^0_0 = -\Gamma^l_{0l,0} - \Gamma^m_{0l}\Gamma^l_{0m} = -\Gamma^\alpha_{0\alpha,0} - \Gamma^\beta_{0\alpha}\Gamma^\alpha_{0\beta}. \tag{40}$$

Taking into account that

$$\Gamma^\alpha_{0\beta} = \frac{g^{\alpha n}}{2}\left(g_{0n,\beta} + g_{\beta n,0} - g_{0\beta,n}\right) = \frac{g^{\alpha n}}{2}g_{\beta n,0} = \frac{\dot{a}}{ca}\frac{g^{\alpha n}}{2}g_{\beta n} = \frac{\dot{a}}{ca}\delta^\alpha_\beta. \tag{41}$$

Thus

$$R^0_0 = -\left[\frac{d}{cdt}\left(\frac{\dot{a}}{ac}\right)\delta^\alpha_\alpha + \left(\frac{\dot{a}}{ac}\right)^2\delta^\beta_\alpha\delta^\alpha_\beta\right] = -\frac{3}{c^2}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right] = -\frac{3\ddot{a}}{ac^2}. \tag{42}$$

Hence

$$-\frac{3\ddot{a}}{ac^2} = \frac{4\pi G}{c^2}(\rho + \frac{3P}{c^2}) \tag{43}$$

and

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + \frac{3P}{c^2})a. \tag{44}$$

This is the relativistic acceleration equation. Compare with

$$\ddot{a} = -\frac{4\pi G\rho}{3}a \tag{45}$$

in the Newtonian theory.