Chapter 20. Relativistic Cosmological Models and Content of the Universe.

CONTENT

20.1. The Energy Conservation Equation

The acceleration equation plus the equation of state contain three variables to find: the scale factor R (or a), the energy density ε ($\varepsilon = \rho c^2$) and pressure P (when $P < 0$ it is better to consider P as a tension). Hence we need one more equation. Let us take as the third equation

$$T^i_{0;i} = 0, \tag{1}$$

which, as was shown in the previous lecture is the consequence of the EFEs and the Bianchi identity. Using the rules of the covariant differentiation we have

$$T^i_{0;i} = T^i_{0,i} + \Gamma^i_{in} T^n_0 - \Gamma^n_{i0} T^i_n = T^0_{0,0} + \Gamma^\alpha_{\alpha 0} T^0_0 - \Gamma^\alpha_{\beta 0} T^\beta_\alpha =$$

$$= \frac{1}{c}\dot{\epsilon} + \frac{\dot{R}}{cR}\delta^\alpha_\alpha \epsilon - \frac{\dot{R}}{cR}\delta^\alpha_\beta T^\beta_\alpha = \frac{1}{c}\left(\dot{\epsilon} + \frac{3\dot{R}}{R}\epsilon + \frac{\dot{R}}{R}\delta^\alpha_\beta \delta^\beta_\alpha P\right) =$$

$$= \frac{1}{c}\left[\dot{\epsilon} + \frac{3\dot{R}}{R}(\epsilon + P)\right] = 0. \tag{2}$$

Hence,

$$\dot{\rho} = -\frac{3\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right). \tag{3}$$

This is the energy conservation equation. If $P = 0$ this equation gives the equation of mass conservation for dust. It is interesting to mention that this equation is identical to the first law of thermodynamics which says that the change of energy ,$E = \rho c^2 V$, in volume, V, surrounded by the surface of area S is equal to the work done by the pressure forces, $F_P = SP$,

$$dE = -F_P dx = -PSdx = -PdV, \text{ or } \dot{E} = -P\dot{V}, \tag{4}$$

Let us consider a sphere of radius r, in this case

$$V = \frac{4\pi}{3} r^3, \tag{5}$$

and

$$\left(r^3 \rho\right)^{\cdot} = -\frac{P}{c^2} \left(r^3\right)^{\cdot}, \tag{6}$$

then

$$\dot{\rho} r^3 + 3r^2 \dot{R} \rho = -\frac{P}{c^2} 3r^2, \tag{7}$$

from which we have

$$\dot{\rho} = -\frac{3\dot{r}}{r} \left(\rho + \frac{P}{c^2}\right). \tag{8}$$

Taking into account that in the homogeneously expanding Universe

$$\frac{\dot{r}}{r} = \frac{\dot{R}}{R}, \tag{9}$$

we obtain exactly the energy conservation equation derived from the EFEs and the pure geometrical Bianchi identity. It means that the first law of thermodynamics is a consequence of the EFEs! Actually we completed the required set of equations describing the relativistic cosmological models. Now we are going to derive from these two equations, the acceleration equation and the energy conservation equation, the relativistic version of the Friedman equation.

20.2. The Friedman Equation for for FLRW metric. Great Surprise.

From the energy conservation equation we obtain

$$\frac{P}{c^2} = -\rho - \frac{\dot{\rho} R}{3\dot{R}}. \tag{10}$$

Putting this expression for p into the acceleration equation, we have

$$\ddot{R} = -\frac{4\pi GR}{3}\left(\rho + \frac{3P}{c^2}\right) = -\frac{4\pi GR}{3}\left(\rho - 3\rho - \frac{\dot{\rho}R}{\dot{R}}\right) =$$

$$= \frac{4\pi G}{3\dot{R}}\left(2\rho R\dot{R} + \dot{\rho}R^2\right) = \frac{4\pi G}{3\dot{R}}\left(\rho R^2\right)^{\cdot}. \tag{11}$$

then multiplying both sides of this equation by $2\dot{R}$ and taking into account that

$$2\dot{R}\ddot{R} = \left(\dot{R}^2\right)^{\cdot}, \tag{12}$$

we obtain

$$\left(\dot{R}^2\right)^{\cdot} = \frac{8\pi G}{3}\left(\rho R^2\right)^{\cdot}, \tag{13}$$

hence

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2, \tag{14}$$

This is the Friedmann equation in the relativistic Cosmology [k appears here as a dimensionless integration constant, exactly as in the Newtonian cosmological models, but as one can easily show is related now with 3-curvature].

This is rather surprising result: the Friedman equation is identical to the equation obtained in the Newtonian theory and does not contain the pressure term at all. However, the pressure is really extremely important. Indeed, if we put the equation of state

$$P = \alpha\rho c^2, \tag{15}$$

into the energy conservation equation we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+\alpha)\frac{\dot{R}}{R}, \tag{16}$$

hence

$$(\ln\rho)^{\cdot} + 3(1+\alpha)(\ln R)^{\cdot} = [\ln\rho + 3(1+\alpha)\ln R]^{\cdot} = \left\{\ln[\rho R^{3(1+\alpha)}]\right\}^{\cdot} = 0, \tag{17}$$

thus

$$\ln[\rho R^{3(1+\alpha)}] = C \tag{18}$$

and

$$\rho R^{3(1+\alpha)} = C'. \tag{19}$$

Finally we obtain that

$$\rho = \rho_0 \left(\frac{R_0}{R}\right)^{3(1+\alpha)}. \tag{20}$$

If

$$\frac{8\pi G\rho}{3} \gg \frac{|k|c^2}{R^2}, \tag{21}$$

the Friedman equation is reduced to

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3}. \tag{22}$$

Substituting into this equation the expression for ρ obtained above, we have

$$\frac{\dot{R}}{R} = \left(\frac{8\pi G\rho}{3}\right)^{1/2} \left(\frac{R_0}{R}\right)^{\frac{3(1+\alpha)}{2}}; \tag{23}$$

we can solve this equation by the separation of variables. For that let introduce

$$x = \frac{R}{R_0}\ \beta = \frac{3(1+\alpha)}{2} \text{ and } A = \left(\frac{8\pi G\rho}{3}\right)^{1/2}. \tag{24}$$

In terms of x, β and A the above equation can be written as

$$\dot{x}x^{\beta-1} = A, \tag{25}$$

then

$$\frac{1}{\beta}dx^{\beta} = Adt, \tag{26}$$

hence

$$x^{\beta} = \beta At + C, \tag{27}$$

since $x = 0$ at $t = 0$ we put $C = 0$. Thus

$$x^{\beta} \sim t, \tag{28}$$

and

$$R \sim x \sim t^{\frac{1}{\beta}} = t^{\frac{2}{3(1+\alpha)}}. \tag{29}$$

This solution is valid if

$$\frac{8\pi G\rho}{3} \gg \frac{|k|c^2}{R^2}, \tag{30}$$

the LHS goes like $R^{-3(1+\alpha)}$ while the RHS goes like R^{-2}. Hence if $-3(1+\alpha) > -2$ or $\alpha < -1/3$ our solution is valid for small R, if $\alpha > -1/3$ our solution is valid for large R.

20.3. Inflation and Expansion with Acceleration

Let the scale factor depends on time as

$$R \sim t^{\gamma}. \tag{31}$$

Taking into account that the physical distances between any two remote objects in the Universe are proportional to R, while the cosmological horizon is proportional to t, it is clear that if $\gamma > 1$ sooner or later any two objects will be outside the cosmological horizon, i.e. will be causally disconnected. Such fast expansion of the Universe is called inflation. On other hand, the condition that the Universe is expanding with acceleration rather than with deceleration,

$$\ddot{R} \sim \gamma(\gamma - 1)t^{\gamma-2} > 0, \tag{32}$$

means that if $\gamma > 1$ the universe expands with acceleration. In order words we can say that the inflation always implies the expansion with acceleration.
From the previous section we know that γ is related to the equation of state parameter α as

$$\gamma = \frac{2}{3(1+\alpha)}. \tag{33}$$

Thus, the condition of inflation can be written as

$$\frac{2}{3(1+\alpha)} > 1, \quad \alpha < -\frac{1}{3}. \tag{34}$$

If $\alpha = -1/3$ and $\gamma = 1$ the acceleration od expansion is equal to zero. Such situation corresponds to domination of cosmic strings. If $\alpha = -1$ corresponds to domination of dark energy in the form of Λ-term. Indeed, in this case, the stress-energy tensor is

$$T_{ik} = (\epsilon + P)u_i u_k - g_{ik}P = \epsilon g_{ik} \tag{35}$$

and as follows from the conservation of energy equation

$$\dot{\epsilon} = -\frac{3\dot{R}}{R}(\epsilon + P) = 0, \tag{36}$$

which means that $\epsilon = \text{const}$, hence T_{ik} has the same form as the Λ-term, i.e. const g_{ik}. In this case $\gamma \to \infty$ and formally we can not apply the power low to the evolution of the scalar parameter and should consider this case separately. As follows from the Friedman equation for spatially flat Universe ($k = 0$) the Hubble constant

$$H = \frac{\dot{R}}{R} = \sqrt{\frac{8\pi G\epsilon}{3c^2}} = \text{const}, \tag{37}$$

Hence the scale factor satisfies the following equation

$$\dot{R} - HR = 0 \tag{38}$$

and the obvious solution of this equation is

$$R \sim e^{Ht}. \tag{39}$$

This is the exponential inflation.

20.4. The Hubble constant as a function of the scale factor

The real Universe contains different forms of matter, radiation and dark energy. Let us call them components. At different epochs the different components dominate. Let us consider the situation when there are several not interacting with each other components. Let ascribe to each component some number $i = 1, 2, 3...N$. Each component is characterized by its own equation of sate

$$P_{(i)} = \alpha_{(i)}\rho_{(i)}c^2. \tag{40}$$

The density of each component evolves according to its own law

$$\rho_{(i)}(t) = \rho_{(i)}(t_0)\left(\frac{R}{R_0}\right)^{-3(1+\alpha_{(i)})} = \rho_{cr}\Omega_{(i)}\left(\frac{R}{R_0}\right)^{-3(1+\alpha_{(i)})}, \tag{41}$$

where

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} \tag{42}$$

is the critical density at the present moment and $\Omega_{(i)}$ is the dimensionless density parameter taken also at the present moment. According to the Friedman equation for spatially flat universe

$$H^2 = \frac{8\pi G\rho}{3}, \tag{43}$$

where

$$\rho = \Sigma_{i=1}^{N}\rho_{(i)}(t) = \rho_{cr}\Sigma_{i=1}^{N}\Omega_{(i)}\left(\frac{R}{R_0}\right)^{-3(1+\alpha_{(i)})}. \tag{44}$$

Hence

$$H = H_0 \sqrt{\Sigma_{i=1}^{N} \Omega_{(i)} \left(\frac{R}{R_0} \right)^{-3(1+\alpha_{(i)})}}. \tag{45}$$

It is nice looking at this formula and Fig.1.0 to think about the history of the Universe with the help of relativistic cosmological model.