Lecture 6. Examples, problems and summary for Part I.

CONTENT

A G Polnarev. Mathematical aspects of cosmology (MAS347), 2009. Week 2. PART I. A non Mathematical Introduction. Lecture 6. Examples, problems and summary for Part I. 6.1. Example for the Lecture 1. The cosmological principle. 6.2. Example for Lecture 2. What is the distance between dark objects?

6.1. Example for Lecture 1. The cosmological principle

QUESTION: (i) Explain briefly what is meant by the cosmological principle and (ii) what observational data support it. (iii) What is the approximate scale at which homogeneity is reached?

ANSWER: (i) The cosmological principle says: The Universe is the same everywhere.

(ii) The expansion of the Universe, the evolution of objects in the Universe and the isotropy of Cosmic Microwave Background support the cosmological principle.

(iii) This scale is about 100 Mpc.

6.2. Example for Lecture 2. What is the distance between dark objects?

PROBLEM: Assume that a small fraction of the matter density in the Universe can be explained by some hypothetical dark objects of mass

$$M = 10^{-3} M_{\odot},$$

where

$$M_{\odot} = 2 \times 10^{30}\ \mathrm{kg}$$

is solar mass (these objects are as massive as the Jupiter). The contribution of these objects to the density of the Universe is

$$\rho_{obj} = 10^{-34}\ \mathrm{kg\ m}^{-3}.$$

Estimate the average distance between these objects.

SOLUTION: The number density of these objects is

$$n_{obj} = \frac{\rho}{M_{obj}}, \tag{1}$$

and the average distance between these objects is determined from

$$d_{obj}^3 n_{obj} \approx 1, \tag{2}$$

hence (see Fig.6.1.)

$$d_{obj} = n_{obj}^{-1/3} = \left(\frac{M_{obj}}{\rho}\right)^{1/3} = \left(\frac{10^{-3} M_{\odot}}{10^{-34}\ \mathrm{kg\ m}^{-3}}\right)^{1/3} = \left(\frac{10^{-3} \times 2 \times 10^{30}\ \mathrm{kg}}{10^{-34}\ \mathrm{kg\ m}^{-3}}\right)^{1/3} \tag{3}$$

Finally,

$$d_{obj} = \left(2 \times 10^{61}\right)^{1/3}\ \mathrm{m} \approx 3 \times 10^{20}\ \mathrm{m} \approx 10\ \mathrm{kpc}. \tag{4}$$

6.3. Example for Lecture 3. Hubble law in vector form

PROBLEM: (i) Write down the Hubble law in vector form.

(ii) Consider three galaxies in an expanding Universe located at points a, b and c (see Fig.6.2.). Prove that if the Hubble law is valid for an observer at a, then it is also valid for observers at b and at c.

(iii) Assume that the vector $\vec{r}_{ab}$ is perpendicular to the vector $\vec{r}_{ac}$. For an observer in the galaxy a the galaxy b is redshifted with $z^b_{(a)} = 0.6$ and the galaxy c is redshifted with $z^c_{(a)} = 0.8$. Find the redshift $z^c_{(b)}$ of the galaxy c, measured by an observer in the galaxy b.

SOLUTION:

$$\text{(i) The Hubble law in vector form is } \vec{v} = H\vec{r}. \tag{5}$$

(ii) Using this vector form, we have for the observer in a

$$\vec{v}_{ab} = H\vec{r}_{ab}, \quad \vec{v}_{ac} = H\vec{r}_{ac}, \tag{6}$$

Hence for the observer in b we have

$$\vec{v}_{ba} = -\vec{v}_{ab} = -H\vec{r}_{ab} = H\vec{r}_{ba}, \tag{7}$$

$$\text{and } \vec{v}_{bc} = \vec{v}_{ba} + \vec{v}_{ac} = -\vec{v}_{ab} + \vec{v}_{ac} = -H\vec{r}_{ab} + H\vec{r}_{ac} = H(-\vec{r}_{ab} + \vec{r}_{ac}) = H(\vec{r}_{ba} + \vec{r}_{ac}) = H\vec{r}_{bc}. \tag{8}$$

Then, for the observer c we have

$$\vec{v}_{ca} = -\vec{v}_{ac} = -H\vec{r}_{ac} = H\vec{r}_{ca} \text{ and } \vec{v}_{cb} = -\vec{v}_{bc} = -H\vec{r}_{bc} = H\vec{r}_{cb}. \tag{9}$$

(iii) Redshif z is related with velocity as

$$z = \frac{v}{c} = \frac{Hr}{c}, \tag{10}$$

Hence

$$\left[z^c_{(b)}\right]^2 = \left(\frac{H}{c}\right)^2 [r_{cb}]^2 = \left(\frac{H}{c}\right)^2 \left[[r_{ca}]^2 + [r_{ab}]^2\right] = \left(\frac{H}{c}\right)^2 [r_{ac}]^2 + \left(\frac{H}{c}\right)^2 [r_{ab}]^2 = \left[z^c_{(a)}\right]^2 + \left[z^b_{(a)}\right]^2. \tag{11}$$

$$\text{Finally } z^c_{(b)} = \sqrt{(0.8)^2 + (0.6)^2} = \sqrt{0.64 + 0.36} = 1. \tag{12}$$

6.4. Example for Lecture 4. Olber's paradox in a static Universe

PROBLEM: (i) Show that in an infinite static Universe with a uniform density of sources the night sky should be as bright as the Sun (the Olber's paradox).

(ii) Explain qualitatively why the evolution and expansion of the Universe can resolve this paradox.

SOLUTION: (i) In an infinite static Universe with a uniform density of sources n, the number of sources per steradian in a shell of radius r and thickness dr centered around an observer at O is (seeFig.6.3.)

$$dN = nr^2 dr. \tag{13}$$

The total intensity from all the sources with luminosity P within radius R is

$$I = n \int_0^R r^2 P/r^2 dr = nPR. \tag{14}$$

Formally this diverges as $R \to \infty$. More accurately, sources would cover the sky when $R \sim (nr_s^2)^{-1}$ and one would expect the night sky should be as bright as the Sun. This is the paradox.

(ii) The Olber's paradox can be resolved partly due to evolution, since the Universe has a finite age and there are no sources more distant than the Hubble scale, partly due to expansion of the Universe, since the flux from the sources is reduced by redshift.

6.5. Useful questions to think about on your own (for lecture 5)

QUESTIONS:

(i) The ionization energy of hydrogen is $E = 13.6$eV. What is the corresponding temperature?

(ii) Explain qualitatively why hydrogen is fully ionized even when the temperature is considerably smaller than E.

(iii) Why is the epoch of recombination also called the decoupling epoch?

(iv) Explain how the anisotropy of the microwave background yields information about the physical conditions in the universe at the moment of decoupling.

6.6. Summary

(i) The Universe is homogeneous (cosmological principle) at scales about 100 Mpc.

(ii) The Universe is expanding according to Hubble law.

(iii) The Universe at the present epoch is expanding with acceleration, which means that dynamically it is dominated by dark energy.

(iv) The structure of Universe is dominated by cold dark matter.

(v) The number density of CMB photons is about 10^9 larger than the number density of barions.

(vi) CMB is the main source of information about the past of the Universe.