PART II. Newtonian Cosmological Models

Lecture 7. Homogeneity and Hubble Law

CONTENT

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7.1. Lagrangian coordinates versus Eulerian coordinates

The Eulerian reference frame is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows. [This can be visualized by sitting on the bank of a river and watching the water pass the fixed location.]

The Lagrangian reference frame (or the co-moving reference frame) is a way of looking at fluid motion where the observer follows individual fluid particles as they move through space and time. Plotting the position of an individual particle through time gives the pathline of the particle.[This can be visualized by sitting in a boat and drifting down a river.]

The Lagrangian coordinates are like "labels" for all objects, particles in the Universe or elements of fluid, i.e. when given to some object once this label is associated with this object for ever (like numbers on shirts of football players). In other words, Lagrangian coordinates are constants for any selected object. In other words, the fundamental difference between Lagrangian and Eulerian coordinates is that for an arbitrary moving object the Lagrangian coordinates are constants, while the Euilerian coordinates could be arbitrary and rather complicated functions of time.

7.2. Lagrangian coordinates in the case of spherically symmetric motion

Let us choose spherical coordinates with the center at the point of our location. The position of any physical object or any point in the Universe can be characterized by some radius-vector corresponding to Eulerian coordinates, $\vec{r}$. In terms of Lagrangian coordinates, which we denote as $\vec{\chi}$, the radius-vector of any selected element is

$$\vec{r} = \vec{r}(t, \vec{\chi}) \equiv \vec{r}(t, \chi_1, \chi_2, \chi_3). \tag{1}$$

Taking into account that all the motions we consider in this section are radial, we can say that

$$\vec{r}(t, \vec{\chi}) = \Phi(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}), \tag{2}$$

where t_0 is an arbitrarily chosen initial moment of time and $\Phi(t, \vec{\chi})$ is a scalar function. Then the velocity of selected element is

$$\vec{v}(t, \vec{\chi}) = \frac{\partial \Phi(t, \vec{\chi})}{\partial t}\vec{r}(t_0, \vec{\chi}) \equiv \dot{\Phi}(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}). \tag{3}$$

7.3. Hubble law and scale factor

If we put Eqs (2) and (3) into the Hubble law in the vector form, $\vec{v} = H\vec{r}$, we obtain

$$\dot{\Phi}(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}) = H(t)\Phi(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}), \tag{4}$$

which after obvious contraction is reduced to a scalar equation

$$\dot{\Phi}(t, \vec{\chi}) = H(t)\Phi(t, \vec{\chi}). \tag{5}$$

Now we can obtain a general solution of this equation, indeed

$$\frac{\dot{\Phi}(t, \vec{\chi})}{\Phi(t, \vec{\chi})} = H(t), \tag{6}$$

hence

$$[\ln \Phi(t, \vec{\chi})]^{\cdot} = H(t), \tag{7}$$

$$\ln \Phi(t, \vec{\chi}) = \int_{t_*}^{t} H(t')dt' + \Psi(\vec{\chi}), \tag{8}$$

where t_* is an arbitrary moment of time and $\Psi(\vec{\chi})$ is an arbitrary function of Lagrangian coordinates $\vec{\chi}$ only, i.e. Ψ does not depend on time. We can rewrite Eq. (8) as

$$\Phi(t, \vec{\chi}) = \exp\left[\int_{t_*}^{t} H(t')dt' + \Psi(\vec{\chi})\right] = R(t)f(\vec{\chi}), \tag{9}$$

where

$$f(\vec{\chi}) = \exp\left[\Psi(\vec{\chi})\right], \tag{10}$$

is also arbitrary function of $\vec{\chi}$ only, while

$$R(t) = \exp\left[\int_{t_*}^{t} H(t')dt'\right] \tag{11}$$

is a function of time only and does not depend on Lagrangian coordinates.

Substituting Eq.(9) into Eq. (5) and then into Eq. (2), we obtain correspondingly

$$\frac{\dot{R}(t)}{R(t)} = H(t), \tag{12}$$

and

$$\vec{r}(t, \vec{\chi}) = R(t)\vec{q}(\vec{\chi}), \tag{13}$$

where

$$\vec{q}(\vec{\chi}) = f(\vec{\chi})\vec{r}(t_0, \vec{\chi}). \tag{14}$$

Eq.14 can be considered as a transformation from one set of Lagrangian coordinates $\vec{\chi} \equiv (\chi_1, \chi_2, \chi_3)$ to another set of Lagrangian coordinates $\vec{q} \equiv (q_1, q_2, q_3)$. Hence, we can forget about $\vec{\chi}$ and express $\vec{r}$ in terms of $\vec{q}$ as

$$\vec{r}(t, \vec{q}) = R(t)\vec{r}(t_0, \vec{q}). \tag{15}$$

Eq. 12 on other hand can be considered as the definition of $R(t)$ in terms of the Hubble constant $H(t)$. And vice verse, if $R(t)$ is known we can calculate $H(t)$ using this equation. As follows from Eq. 15 it is reasonable to call $R(t)$ as the scale factor which describes the Universe as a whole because it does not depend on Lagrangian coordinates.

7.4. What is a cosmological model?

Let us assume that we labeled all elements of the Universe by Lagrangian coordinates q and know the scale factor, $R(t)$, at some moment of time, t_0. As follows from Eq. 15, this means that me know Euler positions of all these elements at this moment of time as well. Hence, if we knew the dependence of function $R(t)$ on time we would be able to reconstruct positions of all elements of the Universe in the past and predict these positions in the future. Then using the Hubble law we can obtain all Hubble velocities at any moment of time.

Let me remind you that such a description of the Universe implies an averaging over scales of order 100 Mpc's. Only after such averaging we can consider the Universe as homogeneous. It is obvious that any peculiar velocities can not be obtained in such a way, because in order to calculate them we need more information about the structure of the Universe and the physical properties of different objects. But at this stage we don't need to care about the peculiar velocities at all, because the averaging over scales of order 100 Mpc gives zero average for the peculiar velocities.

Let me remind you also that when we say that H is called Hubble's constant, we mean that it is the same everywhere in the Universe and depends on time only. This is the homogeneity of the Universe, which obviously implies that all length scales change in the same way.

Homogeneity also requires that the fluid density be the same everywhere at a given time and depends only on time. This statement actually is a definition of cosmic time: one can measure density to determine time.

Thus we conclude that in order to describe the Universe as whole it is enough to determine the scale factor as a function of time. To do this we should write a complete set of mathematical equations which enable us to determine $R(t)$. Such a set of equations is called a cosmological model. The required equations come from physics. Hence the set of equations we use for construction of the cosmological model depends on the physical theory we use and assumptions we made within the framework of the chosen physical theory. In the next lecture we will work with Newtonian theory of gravity to obtain the Newtonian cosmological model. Later, after some introduction to General Relativity (GR) we will have obtained the Relativistic cosmological models based on GR.