A G Polnarev. Mathematical aspects of cosmology (MTH6123), 2009. Week 3. PART II. Newtonian Cosmological Models. Lecture 8. Acceleration equation and conservation of mass.

Lecture 8. Acceleration equation and conservation of mass

CONTENT

8.1. Newtonian gravity and homogeneous sphere of dust

The fact that the Universe is isotropic and homogeneous (the Cosmological Principle) leads to a very simple set of cosmological models. In this part we derive these models in the context of Newtonian theory and in the next part we will use General Relativity. The Newtonian approach is not self-consistent and can not explain the expansion with acceleration (as will be shown later), however, it gives correct results for the evolution of the Universe when the pressure of matter is negligible. Such kind of matter is called dust in cosmology.

Let us now consider the equation of motion for a particle of mass Δm on the surface of the sphere with radius r and mass M (see Fig.8.1 . It is easy to prove that the gravitational field of a uniform medium external to a spherical cavity is zero (for more detail see Lecture 11). According to the Newtonian theory of gravity the gravitational field generated by any spherically symmetric distribution of mass within radius r is the same as if all mass was concentrated at the center. The second law of Newton combined with Newton's gravity law therefore implies that the equation of motion for the particle is

$$\Delta m \ddot{\vec{r}} = -\frac{GM\Delta m \vec{r}}{r^3}, \tag{1}$$

hence for the radial acceleration we obtain

$$\vec{a} \equiv \ddot{\vec{r}} = -\frac{GM\vec{r}}{r^3}. \tag{2}$$

We can see that Δm canceled. This is related to the fact that inertial and gravitational masses of any body are equal to each other (for more detail see Lecture 13).

Homogeneity of the Universe requires that the fluid density is the same everywhere at a given time, so that

$$\rho(\vec{r}, t) = \rho(t). \tag{3}$$

Hence the mass of the sphere is

$$M = \frac{4\pi\rho r^3}{3}. \tag{4}$$

Substituting Eq. 8.4 into Eq. 8.2 we obtain

$$3\vec{a} = -4\pi G\rho(t)\vec{r}. \tag{5}$$

Now we can use the expression for $\vec{r}$ in terms of time, t, and Lagrangian coordinates, $\vec{q}$ (see Eq.2 from Lecture 7)

$$\vec{r}(t,q) = R(t)\vec{q} \tag{6}$$

to calculate the acceleration

$$\vec{a} \equiv \ddot{\vec{r}} = \ddot{R}(t)\vec{q}. \tag{7}$$

Finally, substituting Eq. 6 and Eq. 7 into Eq. 5 and taking into account that the Lagrangian vector q cancels, we obtain the following equation for the scale factor

$$3\ddot{R}(t) = -4\pi G\rho(t)R(t). \tag{8}$$

This is the so called acceleration equation. We can see that according to Newton's theory $\ddot{R}(t)$ is always negative because the density, $\rho(t)$ is always positive. As follows from the Hubble law (Eq. 6 of Lecture 3)

$$\vec{v} = H\vec{r}, \tag{9}$$

and the definition of scale factor (Eq. 12 of Lecture 7)

$$\frac{\dot{R}}{R} = H, \tag{10}$$

all Hubble velocities,

$$\vec{v}(t,\vec{q}) = H(t)\vec{r}(t,\vec{q}) = \frac{\dot{R}}{R} \cdot R\vec{q}, \tag{11}$$

are proportional to $\dot{R}$. Since $\ddot{R} < 0$ the derivative of the scale factor, $\dot{R}$, decays with time, which means that all Hubble velocities decrease in the course of expansion of the Universe. That is why we say that the Universe, according to the Newtonian theory prediction, should always expand with deceleration. The acceleration equation contains two unknown functions, $R(t)$ and $\rho(t)$, and can not be solved unless we supplement it with another equation relating $R(t)$ and $\rho(t)$. This supplementary equation can be obtained from the mass conservation law.

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8.2. Conservation of mass

We assume that the work done by forces of pressure is negligible, otherwise the change in the internal energy of the fluid would also contribute to the change in M. This assumption is correct if all "particles" are non-relativistic, so that their total energy is dominated by their rest-mass energy. This means that mass is conserved, i.e.

$$M = \frac{4\pi\rho(t)r^3}{3} = \text{constant}; \tag{12}$$

which means that

$$\rho(t) = \rho(t_0)\frac{r^3(t_0,\vec{q})}{r^3(t,\vec{q})} = \rho_0\frac{R^3(t_0)q^3}{R^3(t)q^3} = \rho(t_0)\frac{R^3(t_0)}{R^3(t)}. \tag{13}$$

This is the conservation of mass equation.
In the above equations the index "$_0$" corresponds to some arbitrary moment of time t_0. In this course we will refer "$_0$" to the present moment or, in other words, to the moment of observations.

8.3. The final set of equations for Newtonian cosmological model

Now we have the following set of three differential equations

$$H(t) = \frac{\dot{R}(t)}{R(t)}, \tag{14}$$

$$\ddot{R}(t) = -\frac{4\pi G}{3}\rho(t)R(t), \tag{15}$$

$$\rho(t) = \frac{A}{R^3(t)}, \tag{16}$$

where

$$A = \rho(t_0)R^3(t_0). \tag{17}$$

These three equations contain three unknown functions, R, H and ρ and can be solved. We call these equations a cosmological model because R, H and ρ are the same everywhere and are fundamental quantities describing expansion of the Universe as a whole.
Substituting (iii) into (ii) we obtain the following equation for $R(t)$

$$\ddot{R} = -\frac{4\pi G\rho(t_0)R^3(t_0)}{3R^2}. \tag{18}$$

In the next lecture we will try to solve this equation.