Lecture 9. Friedman equation and cosmological parameters

CONTENT

9.1. Derivation of Friedman equation

The acceleration equation is the second order differential equation. To integrate this equation one can multiply both sides of this equation by $\dot{R}$. Taking into account that

$$\ddot{R}\dot{R} = \frac{1}{2}\frac{d\dot{R}^2}{dt} \tag{1}$$

$$\frac{-\dot{R}}{R^2} = \frac{dR^{-1}}{dt}, \tag{2}$$

We obtain

$$\frac{d}{dt}\left(\dot{R}^2 - \frac{8\pi G A}{3R}\right) = 0, \tag{3}$$

Hence,

$$\frac{1}{2}\dot{R}^2 = \frac{4\pi G}{3}AR^{-1} + C = \frac{4\pi G\rho R^2}{3} - \frac{1}{2}kc^2, \tag{4}$$

where C is an integration constant which is expressed for convenience as $C = -kc^2/2$.

The first two terms look like the kinetic and potential energies per unit mass, this equation from mathematical point of view equivalent to the Newtonian energy equation with the integration constant representing the total energy per unit mass.

However, R is the scale factor rather than the radius of any particular sphere. Even dimensions of R are not specified by the above equation. In this course we choose R to have the dimensions of length, in this case k is dimensionless and one can then produce re-scaling of scale factor such that k is 0, 1 or -1. We can write down this equation as

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} - kc^2. \tag{5}$$

This is the Friedman equation.

9.2. Re-scaling of the scale factor

Let me remind you that we have the freedom in our choice of Lagrangian coordinates

$$r(t, \vec{q}) = R(t)q, \tag{6}$$

this means that we can re-scale the Lagrangian coordinate and the scale factor in a self-consistent way:

$$q = \alpha^{-1}\tilde{q}, \quad R = \alpha\tilde{R}, \tag{7}$$

where α is an arbitrary constant. As a result the Eulerian coordinate is unchanged. The Friedman equation can be re-written in terms of the new scale factor as

$$\frac{\alpha^2\dot{\tilde{R}}^2}{\alpha^2\tilde{R}^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{\alpha^2\tilde{R}^2}, \quad \text{or} \quad \frac{\dot{\tilde{R}}^2}{\tilde{R}^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{\alpha^2\tilde{R}^2}. \tag{8}$$

We can define α as follows

$$\begin{array}{cl} \alpha = \sqrt{k}, & \text{if } k > 0 \\ \alpha \text{ is arbitrary} & \text{if } k = 0 \\ \alpha = \sqrt{-k}, & \text{if } k < 0 \end{array} \tag{9}$$

Then we define a new constant of integration

$$\tilde{k} = \frac{k}{\alpha^2}, \tag{10}$$

which is equal to

$$\begin{array}{cl} \tilde{k} = 1, & \text{if } k > 0 \\ \tilde{k} = 0 & \text{if } k = 0 \\ \tilde{k} = -1, & \text{if } k < 0 \end{array} \tag{11}$$

Then we replace back $\tilde{k}$ by k and $\tilde{R}$ by R.

9.3. The most important cosmological parameters

(i) **Hubble parameter,** H_0, **which is defined as the value of Hubble constant at the present moment of time (the time of observations).**

(ii) **Density parameter,** Ω_0, **defined as a ratio**

$$\Omega_0 = \frac{\rho_0}{\rho_{cr}}, \tag{12}$$

where $\rho = \rho(t_0)$ **is the actual average density of the Universe observed at the present moment and**

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} \tag{13}$$

has dimensions of density and is determined by the value of the Hubble parameter.

Hence both these parameters can be determined directly from observations. We introduce also two other parameters which, as we will see in the next section, also can be determined from observations.

(iii) **Deceleration parameter q defined as**

$$q = -\frac{\ddot{R}R}{\dot{R}^2}. \tag{14}$$

We can see that q **is a dimensionless parameter.**

(iv) **k-parameter appeared in the Friedman equation and, as we know from the previous section, can be equal to** 0, -1 **or** $+1$.

There are several other cosmological parameters which correspond predominantly to the content of the Universe and will be discussed in Part IY of our course.

9.4. Friedman equation as a relationship between cosmological parameters

In Cosmology the Friedman equation plays two roles:

1) It gives the law of expansion, $R(t)$ (see the next Lecture.)

2) Written down for the present moment, $t = t_0$, it gives the relationship between the fundamental cosmological parameters:

$$H_0^2 = \frac{8\pi G\rho_0}{3} - \frac{kc^2}{R_0^2}, \tag{15}$$

hence,

$$H_0^2 \left(\frac{8\pi G\rho_0}{3H_0^2} - 1 \right) = \frac{kc^2}{R_0^2}, \tag{16}$$

and finally we obtain

$$k = \left(\frac{H_0 R_0}{c} \right)^2 (\Omega_0 - 1)\,. \tag{17}$$

The relationship between the deceleration parameter q and the density parameter Ω_0 can be obtained from the acceleration equation:

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{4\pi G\rho_0 R \cdot R}{3\dot{R}^2} = \frac{4\pi G\rho_0 R^2}{3\dot{R}^2} = \frac{4\pi G\rho_0}{3H_0^2} =$$

$$= \frac{1}{2}\frac{\rho_0}{\rho_{cr}} = \frac{\Omega_0}{2}. \tag{18}$$

Due to these relationships q and k can also be determined from direct astronomical observations!