## Electron Shielding

In the above laws, there is one additional factor which must be inserted. Surrounding all the ions in the center of a star is a cloud of electrons (and some positrons); the net negative charge of this cloud alters the Coulomb potential each ion sees and thus increases the reaction rate. To estimate the importance of each term, first consider the Boltzmann equation, which describes the relative density of particles. If the statistical weights of two states are the same, then

$$\frac{n_1}{n_0} = e^{-E/kT} \tag{12.1.1}$$

where $E$ is the energy difference between the two states. Now suppose there is a varying electrostatic potential, $\phi(r)$. For normal stars, $Ze\,\phi \ll kT$, so we can write the density of particles, as a function of $\phi$ as

$$n = \bar{n} = e^{-Ze\phi/kT} \approx \bar{n}\left(1 - \frac{Ze\phi}{kT}\right) \tag{12.1.2}$$

If we separate out the ions and electrons, then

$$n_i = \bar{n}_i\left(1 - \frac{Ze\phi}{kT}\right) \qquad n_e = \bar{n}_e\left(1 + \frac{e\phi}{kT}\right) \tag{12.1.3}$$

and the charge density as a function of the potential is

$$\sigma = \sum_i \bar{n}_i\left(Z_i e - \frac{Z_i^2 e^2 \phi}{kT}\right) - \bar{n}_e\left(e + \frac{e^2\phi}{kT}\right) \tag{12.1.4}$$

This can be simplified by noting that the average charge density must be zero, *i.e.*,

$$\sum_i Z_i e\,\bar{n}_i - e\,\bar{n}_e = 0$$

so

$$\sigma = \sum_i -\frac{Z_i^2 e^2 \phi}{kT}\bar{n}_i - \frac{e^2\phi}{kT}\bar{n}_e = -\frac{e^2\phi}{kT}\left(\sum_i Z_i^2 \bar{n}_i + \bar{n}_e\right) \quad (12.1.5)$$

When we substitute in the definition for $n_i$ and $n_e$ via (5.1.2) and (5.1.4), this becomes

$$\sigma = -\frac{e^2\phi}{kT}\left\{\sum_i \rho N_A \left(\frac{Z_i^2 x_i}{A_i}\right) + \sum_i \rho N_A f_i \left(\frac{Z_i x_i}{A_i}\right)\right\}$$

where $x_i$ is the mass fraction and $f_i$ is the ionization fraction. Since the conditions are such that nuclear reactions are occurring, $f_i = 1$, so the charge density finally becomes

$$\sigma = -\frac{e^2\phi}{kT}\,\rho N_A \sum_i \frac{Z_i(Z_i+1)}{A_i} x_i$$

$$= -\frac{e^2\phi}{kT}\,n\,\mu\,\sum_i \frac{Z_i(Z_i+1)}{A_i} x_i \;=\; -\frac{e^2\phi}{kT}\rho N_A\,\zeta \quad (12.1.6)$$

This charge distribution now allows us to solve for the potential explicitly, (at least in the spherically symmetric case) via the Poisson equation

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi\sigma$$

The solution to this equation (which you can check if you like) is

$$\phi = \frac{Ze}{r} e^{-r/r_D} \quad (12.1.7)$$

where $r_D$ is the Debye-Hückel length

$$r_D = \left(\frac{kT}{4\pi e^2 \rho N_A \zeta}\right)^{1/2} \quad (12.1.8)$$

From the above equation, the potential goes to the normal electrostatic potential, *i.e.*, $\phi \to Ze/r$ as $r \to 0$, so the electron shielding only works effectively at large distances, *i.e.*, $r > r_D$. However, most reactions are due to particles near the Gamow peak; the classical separations of these particles are $r_G = Z_a Z_X e^2/E_0$. A comparison shows that

$$\frac{r_D}{r_G} = \left\{ \frac{kT}{4\pi e^2 \rho N_A \zeta} \right\}^{1/2} \Bigg/ \left\{ \frac{Z_a Z_X e^2}{(b\,kT/2)^{2/3}} \right\}$$

$$= 75.3 \left( \frac{A}{Z_a Z_X \zeta^{3/2}} \right)^{\frac{1}{3}} T_6^{7/6}\, \rho^{-1/2} \tag{12.1.9}$$

For all normal stars, $r_D \gg r_G$, which means that electron shielding is not very effective. Electron shielding changes the potential barrier (11.3) slightly, by

$$\tilde{E} = Z_a e\,\phi = \frac{Z_a Z_X e^2}{r} e^{-r/r_D}$$

$$\approx \frac{Z_a Z_X e^2}{r} \left( 1 - \frac{r}{r_D} \right) = \frac{Z_a Z_X e^2}{r} - \frac{Z_a Z_X e^2}{r_D}$$

$$\approx E - E_D \tag{12.1.10}$$

and thus increases the cross sections and reaction rates by $f = e^{E_D/kT}$, with

$$\ln f = \frac{Z_a Z_X e^2}{r_D kT} = \left( \frac{Z_a Z_X e^2}{kT} \right) \left( \frac{4\pi e^2 \rho N_A \zeta}{kT} \right)^{1/2}$$

or

$$\ln f = 0.188\, Z_a Z_X \left( \frac{\zeta \rho}{T_6^3} \right)^{1/2} \tag{12.1.11}$$

For typical stellar densities and temperatures, this correction will be $\sim 10\%$. This "weak screening" approximation, however, does not apply to the high density-low temperature regime of white dwarfs. Under those conditions, the form of the equation will be different.

## Summary for Reactions

When electron shielding is included, the nuclear reaction rate equation becomes

$$r_{aX} = (1+\delta_{aX})^{-1}\ \frac{\rho^2 X_a X_X N_A^2}{A_a A_X}\, f\ \langle\sigma v\rangle \tag{12.2.1}$$

or, in terms of number density

$$r_{aX} = (1+\delta_{aX})^{-1} N_a N_X f \langle\sigma v\rangle \tag{12.2.2}$$

For non-resonant reactions, $\langle\sigma v\rangle$ can be computed directly from $S(E_0)$, $\left(\frac{\partial S}{\partial E}\right)_{E_0}$, $Z_a$, $Z_X$, $A_a$, $A_X$, and $T$, while for resonant reactions, $\langle\sigma v\rangle$ is derived from $(\omega\gamma)$, $E_r$, and $T$.

Now let's simplify the terminology by first assigning

$$\lambda_{aX} = \langle\sigma v\rangle \tag{12.2.3}$$

and then defining the lifetime of species $X$ against reactions with particle $a$ to be

$$\frac{dN_X}{dt} = -r_{aX} = -\frac{N_X}{\tau_a} \tag{12.2.4}$$

The form of this equation is now identical with that associated with radioactivity or other decay phenomena. (Unlike radioactive decay, of course, $\tau_a$ depends on the external environment.) From the equation above,

$$\tau_a(X) \ = \ (1+\delta_{aX})^{-1}\frac{N_X}{r_{aX}} \ = \ (\lambda_{aX} f N_a)^{-1} \tag{12.2.5}$$

Note that the Kronecker delta disappears from the final lifetime calculation. According to (12.2.1), the reaction rate for identical particles has a factor of two in the denominator, but since each reaction destroys two particles, this factor cancels out. Note also, that by (12.2.4) and (12.2.5), the total lifetime of particle $X$ from all reactions is

$$\frac{1}{\tau} = \sum_i \frac{1}{\tau_i} \tag{12.2.6}$$