Applicability of Polytropes

The four equations of stellar structure divide naturally into two groups: one describing the mechanical structure of the star using (2.1.3) and (2.2.9), and the other giving the thermal structure via (2.3.3) and (2.4.4). However, the only contact between the mechanical variables and thermal equations is through the temperature dependence of the equation of state. Under certain circumstances, however, the pressure can become independent of temperature, and only depend on density, *i.e.*,

$$P = K\rho^{\gamma} = K\rho^{1+1/n} \tag{15.1.1}$$

where n is the *polytropic index.* When the equation of state can be written in this form, the calculations of stellar structure simplify enormously.

Decoupling occurs when:

• Stars become supported by electron degeneracy. When this occurs, the pressure is related to the density by

$$P_e = \frac{h^2}{20 m_e m_a \mu_e} \left(\frac{3}{\pi m_a \mu_e} \right)^{2/3} \rho^{5/3} = K\rho^{5/3} \tag{7.3.7}$$

($K = 1.0036 \times 10^{13}\, \mu_e^{-5/3}$ cgs), or

$$P_e = \left(\frac{hc}{8 m_a \mu_e} \right) \left(\frac{3}{\pi m_a \mu_e} \right)^{1/3} \rho^{4/3} = K\rho^{4/3} \tag{7.3.13}$$

($K = 1.2435 \times 10^{15}\, \mu_e^{-4/3}$ cgs) depending on whether the Fermi momentum is relativistic or not. For future reference, note here that the constant K depends only on μ and atomic physics.

• Energy transport comes only from convection, and radiation pressure is negligible. In this case, $P \approx P_{\rm gas}$, and

$$P = \frac{\rho}{\mu m_a} kT \implies T \propto \left(\frac{P}{\rho}\right)$$

When combined with the definition of $\nabla_{\rm ad}$,

$$\left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad} = \nabla_{\rm ad} \implies P \propto T^{1/\nabla_{\rm ad}}$$

this yields

$$P^{1-(1/\nabla_{\rm ad})} \propto \rho^{-1/\nabla_{\rm ad}}$$

or

$$P = K\rho^{1/(1-\nabla_{\rm ad})} \tag{15.1.2}$$

where K is not a function of atomic physics, but instead depends on the star's boundary conditions. Note that for a completely ionized gas, $\nabla_{\rm ad} = 2/5$, which recovers $P \propto \rho^{5/3}$.

• Regions in which the ratio of the gas pressure to the radiation pressure is constant. If we let $\beta = P_{\rm gas}/P$, then for an ideal gas plus radiation pressure equation of state

$$P = \frac{P_{\rm gas}}{\beta} = \frac{\rho}{\mu m_a \beta} kT \tag{15.1.3}$$

and

$$P = \frac{P_{\rm rad}}{1-\beta} = \frac{aT^4}{3(1-\beta)} \tag{15.1.4}$$

If we then use (15.1.4) to substitute for temperature, then equation (15.1.3) becomes

$$P = \frac{\rho k}{\mu m_a \beta}\left(\frac{3P(1-\beta)}{a}\right)^{1/4}$$

$$P^{3/4} = \frac{\rho k}{\mu m_a \beta}\left(\frac{3(1-\beta)}{a}\right)^{1/4}$$

$$P = \left(\frac{3}{a}\right)^{1/3}\left(\frac{k}{\mu m_a}\right)^{4/3}\left(\frac{1-\beta}{\beta^4}\right)^{1/3}\rho^{4/3} = K\rho^{4/3} \tag{15.1.5}$$

where the value of K depends on the importance of radiation pressure.

This last condition comes about naturally if the ratio $\kappa\mathcal{L}/\mathcal{M}$ is constant. To see this, consider that if the star is convective, $P \propto \rho^{1/(1-\nabla_{\rm ad})}$ as above, but if the star is radiative, then $\nabla = \nabla_{\rm rad}$, and

$$\nabla_{\rm rad} = \frac{1}{16\pi cG}\frac{3P}{aT^4}\frac{\kappa\mathcal{L}}{\mathcal{M}} = \frac{P}{T}\frac{dT}{dP} \tag{3.1.6}$$

If we then substitute using

$$P_{\rm rad} = \frac{aT^4}{3} \quad \text{and} \quad dP_{\rm rad} = \frac{4aT^3}{3}dT$$

then

$$\frac{1}{16\pi cG}\frac{P}{P_{\rm rad}}\frac{\kappa\mathcal{L}}{\mathcal{M}} = \frac{P}{dP}\frac{3\,dP_{\rm rad}}{4aT^4} = \frac{P}{dP}\frac{dP_{\rm rad}}{4\,P_{\rm rad}}$$

or

$$\frac{dP_{\rm rad}}{dP} = \frac{1}{4\pi cG}\frac{\kappa\mathcal{L}}{\mathcal{M}} \tag{15.1.6}$$

Thus, if $\kappa\mathcal{L}/\mathcal{M}$ is constant, and the surface pressure is ~ 0, then

$$\int_0^{P_{\rm rad}(r)} dP_{\rm rad} = C\int_0^{P(r)} dP \quad\Longrightarrow\quad \frac{P_{\rm rad}(r)}{P(r)} = C \tag{15.1.7}$$

This is the condition for the Eddington "standard" model.

(Actually, setting $\kappa\mathcal{L}/\mathcal{M}$ constant is not a terrible assumption. Kramer law opacities decrease with temperature as $T^{-3.5}$, while the proton-proton chain gives $\epsilon_n = \mathcal{L}/\mathcal{M} \propto T^{3.5}$ for low mass stars. Thus, $\kappa\mathcal{L}/\mathcal{M}$ is approximately constant, and (15.1.7) is an adequate approximation.)

Polytropic Calculations

To calculate the structure of a polytropic star, begin by assuming hydrostatic equilibrium, and multiplying both sides by r^2/ρ, *i.e.*,

$$\frac{r^2}{\rho}\frac{dP}{dr} = -\frac{r^2}{\rho}\frac{GM}{r^2}\rho = -GM$$

If we then take the derivative (with respect to r) of both sides, and divide by r^2, then we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\frac{G}{r^2}\frac{dM}{dr} = -\frac{G}{r^2}4\pi r^2\rho = -4\pi G\rho \qquad (15.2.1)$$

(Note that this is nothing more than the Poisson equation — if you substitute for pressure using

$$\frac{d\Phi}{dr} = \frac{GM}{r^2} = -\frac{1}{\rho}\frac{dP}{dr}$$

then you recover $\nabla^2\Phi = 4\pi G\rho$ in spherical coordinates.) If you then substitute in the polytropic relation (15.1.1), then

$$\frac{dP}{dr} = K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr}$$

and Poisson's equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left\{\frac{r^2}{\rho}K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr}\right\} = -4\pi G\rho \qquad (15.2.2)$$

To simplify this expression, let's put it in dimensionless form. First, define a variable θ, such that

$$\theta(r) = (\rho/\rho_c)^{1/n} \quad \Longrightarrow \quad \rho = \rho_c \theta^n \tag{15.2.3}$$

Poisson's equation then becomes

$$K\left\{1+\frac{1}{n}\right\}\frac{1}{r^2}\frac{d}{dr}\left\{r^2 \rho_c^{-1+1/n}\theta^{1-n} n\rho_c \theta^{n-1}\frac{d\theta}{dr}\right\} = -4\pi G \rho_c \theta^n$$

or

$$\frac{(n+1)\,K\rho_c^{1+1/n}}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Then define a dimensionless radius, $\xi = r/r_n$, where

$$r_n = \left\{\frac{(n+1)K}{4\pi G}\right\}^{1/2} \rho_c^{(1-n)/2n} = \left\{\frac{(n+1)P_c}{4\pi G\rho_c^2}\right\}^{1/2} \tag{15.2.4}$$

where we have used (15.1.1) to substitute in the central pressure. In this form, the Poisson equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{15.2.5}$$

This is called the Lane-Emden equation; its solution gives the run of dimensionless density θ as a function of the dimensionless radius, ξ. Since it is a second order differential equation, you need two boundary conditions. The first is at the center: from spherical symmetry, the pressure gradient at the center ($\theta = 1$) must be zero. The second condition comes from the surface, ξ_1, where the density should go to zero. So our boundary conditions are

$$\frac{d\theta}{d\xi}(\xi = 0) = 0 \quad \text{(the center)}$$

and

$$\theta(\xi = \xi_1) = 0 \qquad \text{(the surface)}$$

Unfortunately, the Lane-Emden equation does not have an analytic solution for arbitrary values of n. In fact, there are only four analytic solutions. The first is for $n = 0$, which implies $\rho(r) = \rho_c$, or a constant density sphere. For these models,

$$\theta(\xi) = 1 - \frac{\xi^2}{6} \tag{15.2.6}$$

with $\xi_1 = \sqrt{6}$ to satify the boundary condition of $\theta(\xi_1) = 0$. (This can be trivially checked via substition.) The second solution is for the $n = 1$ case, where

$$\theta(\xi) = \frac{\sin \xi}{\xi} \tag{15.2.7}$$

Note here that there are an infinite number of values of ξ_1 for which $\theta(\xi_1) = 0$. However, in practice, $\xi_1 = \pi$, since all other values would give an unrealistic zero density somewhere in the middle of the star. A third solution exists for $n = 5$,

$$\theta(\xi) = \left(1 + \xi^2/3\right)^{-1/2}$$

For this model, θ only goes to zero when $\xi_1 \longrightarrow \infty$, thus stars with $n = 5$ have infinite radius. All models with $n > 5$ have both infinite radius and infinite mass, but the $n = \infty$ solution is noteworthy. From (15.1.1), $n = \infty$ corresponds to $P = K\rho$, which is the isothermal case. If we go back to the Poisson equation (15.2.2) and let $n = \infty$, then the equation becomes

$$\frac{K}{r^2}\frac{d}{dr}\left\{r^2 \frac{d \ln \rho}{dr}\right\} = -4\pi G \rho \tag{15.2.8}$$

If we let $\Psi = -\ln\rho$ and $\xi = (4\pi G/K)^{1/2}$, then this translates to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Psi}{d\xi}\right) = e^{-\Psi} \tag{15.2.9}$$

This is the equation for an isothermal sphere. It has many applications in astrophysics (especially extragalactic astrophysics).

The remaining solutions of the Lane-Emden equation, including that for $n = 3/2$ ($\gamma = 5/3$) and $n = 3$ ($\gamma = 4/3$), must be computed numerically. For the inner part ($\xi < 1$) of the polytrope, this can be done by writing θ out as a power law, *i.e.*,

$$\theta = 1 + a\xi + b\xi^2 + c\xi^3 + d\xi^4 + \ldots \tag{15.2.10}$$

and substituting into the Lane-Emden equation. When this is done, the left side of the equation becomes

$$f(\xi) = 2a\xi^{-1} + 6b + 12c\xi + 20d\xi^2 + \ldots \tag{15.2.11}$$

Meanwhile, the right side of the equation can be evaluated by expanding θ^n as a Taylor series about $\xi = 0$. In other words,

$$f(\xi) = \theta^n = f(0) + f'(0)\xi + \frac{f''(0)\xi^2}{2!} + \ldots$$

where, through (15.2.10),

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= n\,\theta^{n-1}\frac{d\theta}{d\xi} = n\,(1)\,a \\
f''(0) &= n\left\{(n-1)\theta^{n-2}\frac{d\theta}{d\xi}\frac{d\theta}{d\xi} + \theta^{n-1}\frac{d^2\theta}{d\xi^2}\right\} \\
&= n\left\{(n-1)(1)a^2 + (1)2b\right\}
\end{aligned}$$

Thus,

$$-\theta^n = -1 - n\,a\xi - \frac{n(n-1)a^2 + 2b}{2}\xi^2 + \ldots \tag{15.2.12}$$

The coefficients of (15.2.10) can then be found by equating the terms of (15.2.11) and (15.2.12). In other words, since (15.2.12) does not contain a ξ^{-1} term, $a = 0$, and

$$\begin{aligned}
6b &= -1 & \Longrightarrow \quad & b = -\tfrac{1}{6} \\
12c &= -n\,a & \Longrightarrow \quad & c = 0 \\
20d &= -\tfrac{n(n-1)a^2+2b}{2} & \Longrightarrow \quad & d = \tfrac{n}{120}
\end{aligned}$$

Note that all the odd power terms are zero, due to the fact that $a = 0$. This leaves

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{15120}\xi^6 + \ldots$$

The outer regions of the polytrope can now be computed numerically by translating the second-order Lane-Emden equation into two first-order equations with known starting point boundary conditions. In other words, if we write Lane-Emden as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \quad \Longrightarrow \quad \frac{d^2\theta}{d\xi^2} = -\theta^n - \frac{2}{\xi}\frac{d\theta}{d\xi}$$

then

$$z = \frac{d\theta}{d\xi} \qquad \text{and} \qquad \frac{dz}{d\xi} = -\theta^n - \frac{2}{\xi}z \tag{15.2.13}$$

This can be solved in a straightforward manner by starting from the polynomial expansion and using Runge-Kutta integration.

Some key values resulting from this integration are tabulated below.

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$	$\rho_c/\bar{\rho}$
0.0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800.
5.0	∞	1.73205	∞