Homologous Expansion and Contraction

Many problems in stellar structure can be addressed by approximating the expansion (or contraction) of the star as homologous. This is equivalent to saying that the fractional rate of change in the star is a constant, *i.e.*,

$$\frac{\dot{r}}{r} = \frac{\dot{R}}{R} = \text{Constant} \tag{20.1.1}$$

(An example of such a system would be an expanding polytrope, which keeps the same value of n. Since $\mathcal{M} \propto \xi \propto r/r_n$, the mass elements of the star will remain at homologous points; only the scale length r_n changes.)

Although homologous expansion (or contraction) does not often occur (after all, it does require that the whole star act in unison), it is a good first-order approximation for many problems, and is useful for quick analyses. For instance, we can estimate the energy released (or absorbed) when a star homologously contracts (or expands). According to (20.1.1)

$$\frac{d}{d\mathcal{M}} \left(\frac{d \ln r}{dt} \right) = 0 \tag{20.1.2}$$

If we exchange the two derivatives, then

$$\frac{d}{dt} \left(\frac{d \ln r}{d\mathcal{M}} \right) = \frac{d}{dt} \left(\frac{1}{r} \frac{dr}{d\mathcal{M}} \right) = \frac{d}{dt} \left(\frac{1}{4\pi r^3 \rho} \right) = \frac{1}{4\pi r^3 \rho} \left(-\frac{3\dot{r}}{r} - \frac{\dot{\rho}}{\rho} \right)$$

Setting this to zero then implies that for homologous movement

$$\frac{\dot{\rho}}{\rho} = -3 \, \frac{\dot{r}}{r} \tag{20.1.3}$$

Similarly, we can easily compute how the hydrostatic pressure changes for an homologous mass point under expansion. From (2.2.4)

$$\dot{P} = \frac{d}{dt} \int -\frac{G\mathcal{M}}{4\pi r^4} d\mathcal{M} = \int -\frac{G\mathcal{M}}{4\pi} \frac{d}{dt} \left(\frac{1}{r^4} \right) d\mathcal{M} = \int \left(\frac{4\dot{r}}{r} \right) \frac{G\mathcal{M}}{4\pi r^4} d\mathcal{M}$$

which for constant $\dot{r}/r$ implies

$$\frac{\dot{P}}{P} = -4\,\frac{\dot{r}}{r} \tag{20.1.4}$$

Combining (20.1.3) and (20.1.4) with the equation of state for a chemically homogeneous gas

$$\frac{d\rho}{\rho} = \alpha\,\frac{dP}{P} - \delta\,\frac{dT}{T}$$

gives an expression for temperature

$$\frac{\dot{T}}{T} = \frac{3-4\alpha}{\delta}\,\frac{\dot{r}}{r} \tag{20.1.5}$$

Recall now that the energy associated with the gravitational expansion/contraction of a star is

$$\epsilon_g = -c_P\frac{dT}{dt} + \frac{\delta}{\rho}\frac{dP}{dt} \tag{2.3.4}$$

which, through (1.19) is

$$\epsilon_g = -c_P T\left(\frac{\dot{T}}{T} - \nabla_{\mathrm{ad}}\frac{\dot{P}}{P}\right) \tag{2.3.5}$$

Substituting using the equation of homology then gives an estimate for the total amount of energy associated with an expansion or contraction.

$$\epsilon_g = c_P T\left(\frac{4\alpha-3}{\delta} - 4\nabla_{\mathrm{ad}}\right)\frac{\dot{R}}{R} \tag{20.1.6}$$

For an ideal gas with $\alpha = \delta = 1$ and $\nabla_{\mathrm{ad}} = 2/5$,

$$\epsilon_g = -\frac{3}{5}c_P T\frac{\dot{R}}{R}$$

Note that the value is negative; contraction by the star releases energy, while expansion absorbs energy.

The Gravothermal Specific Heat

Another use of dynamic homology involves the calculation of Gravothermal Specific Heat. Two familiar thermodynamic concepts are the specific heat for a system kept at constant pressure (c_P) and the specific heat for a system in a constant volume (c_V). However, for some stellar applications, a more interesting quantity is the specific heat of a system kept in hydrostatic equilibrium. This is called the Gravothermal Specific Heat.

To compute the gravothermal specific heat, c^*, let us pick a region of the star with radius, r_s and assume that, at least locally, any expansion or contraction near that region is homologous. (That is, $dr = r\,dx$, and after expansion, the nearby shells have radii $r + dr = r(1 + dx)$, where dx is the same for all shells.) From (20.1.3) and (20.1.4)

$$\frac{\dot\rho}{\rho} = -3\frac{\dot r}{r} \implies \frac{d\rho}{\rho} = -3\,dx \qquad (20.2.1)$$

and

$$\frac{\dot P}{P} = -4\frac{\dot r}{r} \implies \frac{dP}{P} = -4\,dx \qquad (20.2.2)$$

Hence, if the expansion is homologous

$$\frac{d\rho}{\rho} = \frac{3}{4}\frac{dP}{P} \qquad (20.2.3)$$

Now let's substitute this into the equation of state

$$\frac{d\rho}{\rho} = \alpha\,\frac{dP}{P} - \delta\,\frac{dT}{T} + \varphi\,\frac{d\mu}{\mu} \qquad (3.2.2)$$

If the chemical composition is constant, then

$$\frac{3}{4}\frac{dP}{P} = \alpha\frac{dP}{P} - \delta\frac{dT}{T}$$

which implies

$$\frac{dP}{P} = \frac{4\delta}{4\alpha - 3} \frac{dT}{T} \tag{20.2.4}$$

Now let's use this result in the first law of thermodynamics. Recall that when written in terms of c_P and $\nabla_{\rm ad}$, the first law was

$$dq = c_P T \left[\frac{dT}{T} - \nabla_{\rm ad} \frac{dP}{P} \right] \tag{1.21}$$

For homologous expansion (or contraction), we then have

$$dq = c_P T \left[\frac{dT}{T} - \nabla_{\rm ad} \frac{4\delta}{4\alpha - 3} \frac{dT}{T} \right] = c^* T \, \frac{dT}{T} \tag{20.2.5}$$

where

$$c^* = c_P \left(1 - \nabla_{\rm ad} \frac{4\delta}{4\alpha - 3} \right) \tag{20.2.6}$$

Thus, the gravothermal specific heat acts like any other specific heat: if an amount of heat, dq is added to the system, then the temperature of the system will rise by $dT = dq/c^*$.

Equation (20.2.6) has interesting implications for normal stars. For an ideal gas equation of state ($\alpha = 1$, $\delta = 1$, $\nabla_{\rm ad} = 2/5$), c^* is negative. Thus, if the heat content of a core is perturbed by an amount dq, the temperature of the core immediately *drops,* thereby reducing the production of energy. (If this did not happen, then we would have a thermal runaway!)

The reason stars are stable is that any extra energy that is produced goes into performing PdV work on the surroundings. If heat is added to the center of an (ideal gas) star, $dT < 0$ (by 20.2.5), which implies $dP < 0$ (from 20.2.4), which implies $dx > 0$ (from 20.2.2). Thus the star expands! The extra heat is used to expand the star, and more energy is used in the expansion than was originally added. The star must extract the requisite additional energy from its internal energy.

Note also that under homologous expansion or contraction,

$$\frac{d\rho}{\rho} = -3\,dx \quad \frac{dP}{P} = -4\,dx \quad \frac{dT}{T} = \left(\frac{3-4\alpha}{\delta}\right) dx \tag{20.2.7}$$

Therefore, since dx is a constant, all these ratios are also constants, with

$$\frac{d\rho}{\rho} = \frac{3\delta}{4\alpha - 3}\frac{dT}{T} \tag{20.2.8}$$

and

$$\frac{dP}{P} = \frac{4\delta}{4\alpha - 3}\frac{dT}{T} \tag{20.2.9}$$

Stability in Core Burning

Now let's consider how a star will react to changes in its rate of nuclear reactions. In equilibrium, the energy generated inside a sphere with mass, $\mathcal{M}$, is balanced by the luminosity transported outward, *i.e.*,

$$\epsilon\mathcal{M} - \mathcal{L} = 0$$

Now, let's perturb the energy generation by an amount $d\epsilon$. In general, the dynamical timescales of stars are short, and the thermal timescales relatively long, so that during the perturbation, the star will remain in hydrostatic equilibrium, but be out of thermal balance. Thus

$$\mathcal{M}d\epsilon - d\mathcal{L} = \mathcal{M}\frac{dq}{dt} = \mathcal{M}c^* \frac{dT}{dt} \tag{20.3.1}$$

or, if we divide through by $\mathcal{L} = \epsilon\mathcal{M}$,

$$\frac{d\epsilon}{\epsilon} - \frac{d\mathcal{L}}{\mathcal{L}} = \frac{\mathcal{M}}{\mathcal{L}} c^* \frac{dT}{dt} \tag{20.3.2}$$

Now let us derive an expression for $d\mathcal{L}/\mathcal{L}$ in the case of radiative energy transfer. From (3.1.5)

$$\mathcal{L} = \frac{64\pi^2 ac}{3} \left(\frac{r^4 T^3}{\kappa} \right) \frac{dT}{d\mathcal{M}}$$

so

$$d\mathcal{L} = \frac{64\pi^2 ac}{3} \left\{ \frac{4r^3T^3}{\kappa}\frac{dT}{d\mathcal{M}}dr + \frac{3r^4T^2}{\kappa}\frac{dT}{d\mathcal{M}}dT \right.$$
$$\left. - \frac{r^4T^3}{\kappa^2}\frac{dT}{d\mathcal{M}}d\kappa + \frac{r^4T^3}{\kappa} d\left(\frac{dT}{d\mathcal{M}}\right) \right\}$$

or

$$\frac{d\mathcal{L}}{\mathcal{L}} = 4\frac{dr}{r} + 3\frac{dT}{T} - \frac{d\kappa}{\kappa} + d\left(\frac{dT}{d\mathcal{M}}\right) \Bigg/ \left(\frac{dT}{d\mathcal{M}}\right) \tag{20.3.3}$$

This can be simplified. First, recall that from the definition of homologous movement and from (20.2.7)

$$dx = \frac{dr}{r} = -\frac{\delta}{4\alpha - 3}\frac{dT}{T}$$

Next, consider that the opacity, κ, is itself a function of density and temperature, with $\kappa \propto \rho^p T^q$. Thus

$$\begin{aligned}
\frac{d\kappa}{\kappa} &= (d\ln\kappa)_T + (d\ln\kappa)_\rho \\
&= \left(\frac{d\ln\kappa}{d\ln\rho}\right)_T \left(\frac{d\rho}{\rho}\right) + \left(\frac{d\ln\kappa}{d\ln T}\right)_\rho \left(\frac{dT}{T}\right) \\
&= p\left(\frac{3\delta}{4\alpha - 3}\right)\frac{dT}{T} + q\frac{dT}{T}
\end{aligned}$$

where we have used (20.2.8) in substituting for $d\rho/\rho$. Moreover, under homology dT/T is a constant, so

$$d\left(\frac{dT}{d\mathcal{M}}\right) = d\left(\frac{dT}{T}\frac{T}{d\mathcal{M}}\right) = \frac{dT}{T}\frac{dT}{d\mathcal{M}}$$

Putting this all together, we get

$$\frac{d\mathcal{L}}{\mathcal{L}} = \left[-\frac{4\delta}{4\alpha - 3} + 3 - \frac{3\delta}{4\alpha - 3}p - q + 1\right]\frac{dT}{T}$$

or, still more simply,

$$\frac{d\mathcal{L}}{\mathcal{L}} = \left\{ 4 - q - \frac{\delta}{4\alpha - 3} (4 + 3p) \right\} \frac{dT}{T} \tag{20.3.4}$$

Similarly, we can substitute for $d\epsilon/\epsilon$ using the energy generation coefficients. If we let $\epsilon \propto \rho^{\lambda} T^{\nu}$ and again use (20.2.8) then

$$\begin{aligned} \frac{d\epsilon}{\epsilon} &= \left(\frac{d \ln \epsilon}{d \ln \rho} \right)_T \frac{d\rho}{\rho} + \left(\frac{d \ln \epsilon}{d \ln T} \right)_\rho \frac{dT}{T} \\ &= \lambda \left(\frac{3\delta}{4\alpha - 3} \right) \frac{dT}{T} + \nu \frac{dT}{T} \end{aligned} \tag{20.3.5}$$

If we now substitute this and (20.3.4) into (20.3.2), we get

$$\left[(\nu + q - 4) + \frac{\delta}{4\alpha - 3} (3\lambda + 3p + 4) \right] \frac{dT}{T} = \frac{\mathcal{M}}{\mathcal{L}} c^* \frac{dT}{dt} \tag{20.3.6}$$

or

$$\frac{dT}{dt} = \frac{K}{c^*} \frac{dT}{T} \tag{20.3.7}$$

where

$$K = \frac{\mathcal{L}}{\mathcal{M}} \left[(\nu + q - 4) + \frac{\delta}{4\alpha - 3} (3\lambda + 3p + 4) \right] \tag{20.3.8}$$

Note the implications of this equation. In the case of an ideal gas ($\alpha = \delta = 1$), proton-proton burning ($\nu \sim 5$, $\lambda = 1$), and Kramers opacity ($q = -7/2$, $p = 1$), $K = 7.5\mathcal{L}/\mathcal{M}$. Since c^* is less than zero, a positive temperature perturbation dT/T will result in a negative dT/dt, *i.e.,* a net cooling. As a result, the temperature perturbation will be damped away. The coefficients for CNO burning, electron scattering, and H^- opacity make K even more positive, increasing the stability.

Now consider a non-relativistic degenerate equation of state with helium fusion. From (7.3.7) $P \propto \rho^{5/3}$, which implies $\delta = 0$, and $\alpha = 3/5$. Thus, we have a positive gravothermal specific heat ($c^* = c_P$), and a positive value of K ($\nu + q - 4 \sim 32$). Under these circumstances, a positive heat fluctuation leads to an increase in temperature, and a further increase in the nuclear reaction rate. We have a thermonuclear runaway! Moreover, from (20.2.8),

$$\frac{d\rho}{\rho} = \frac{3\delta}{4\alpha - 3} \longrightarrow 0$$

Consequently, a stellar core can experience a thermonuclear runaway, while keeping the density (and, by implication, the pressure) constant.

Stability in Shell Burning

The analysis of the thermal stability of shell burning stars follows that for core-burning stars, with one exception. With core burning, the mass within a burning region is $\mathcal{M} = \frac{4}{3}\pi r^3 \rho$, and so under homology

$$\frac{d\rho}{\rho} = -3\frac{dr}{r}$$

However, consider the case of a shell of thickness D at a radius r_0 which is defined by the inert core below. In this case

$$\mathcal{M} = 4\pi r^2 D\rho \Longrightarrow \rho = \frac{\mathcal{M}}{4\pi r^2 D}$$

Since r_0 does not change, and the mass in the shell does not change, $dD = dr$, and

$$d\rho = \frac{\mathcal{M}}{4\pi r^2} d\left(\frac{1}{D}\right) = -\frac{\mathcal{M}}{4\pi r^2}\frac{dD}{D^2} = -\rho\,\frac{dD}{D}$$

or

$$\frac{d\rho}{\rho} = -\frac{r}{D}\frac{dr}{r} = -\frac{r}{D}\,dx \qquad (20.3.9)$$

Since the equations of state and of hydrostatic equilibrium do not change, we can now substitute this new expression for $d\rho$ and recalculate c^*. The result is

$$c^*_{\rm shell} = c_P\left(1 - \nabla_{\rm ad}\frac{4\delta}{4\alpha - r/D}\right) \qquad (20.3.10)$$

(Obviously, this is identical to the original expression, except that the constant "3" has been replaced by "r/D" in the denomintor.)

Now let's again consider the equation of thermal stability (20.3.7). Since K is positive, a negative value of c^* implies thermal stability, while a positive value denotes a thermal runaway. As demonstrated above, if the gas conditions are degenerate, $\delta = 0$, and again we have a runaway. However, suppose we are dealing with an ideal gas (which is the relevant condition, since the gas in the shell is outside the core). For core burning stars, an ideal gas equation of state always produces a negative value of c^*, so the star is stable. But according to (20.3.11), if D is small, r/D will be large enough to make c^* positive. Thus, nuclear burning in thin shells is susceptible to thermonuclear runaways.

In essence what is happening is that if D/r is small, a small change in D causes a large change in ρ, but, in absolute terms, a very minor change in dr/r. Thus the layers above, which are in hydrostatic equilibrium, are barely affected, and $dP/P \approx 0$. The effective equation of state is therefore $\rho \sim 1/T$, for which $\alpha = 0$ and $\delta = 1$. Thus, fusion under these conditions can be unstable and runaways, or *thermal pulses,* can occur.