Post Main Sequence Evolution

Since a star's luminosity on the main sequence does not change much, we can estimate its main-sequence lifetime from simple timescale arguments, and the mass-luminosity relation. If $\mathcal{L} \propto \mathcal{M}^\eta$, then

$$\tau_{\rm MS} \propto \frac{\mathcal{M}}{\mathcal{L}} \propto \mathcal{M}^{1-\eta}$$

If we adopt a main-sequence lifetime of the Sun of 10^{10} years, then

$$\tau_{\rm MS} = 10^{10} \left(\frac{\mathcal{M}}{\mathcal{M}_\odot} \right)^{1-\eta} \text{ years}$$

Since $\eta \sim 3.5$, the main-sequence lifetime of a star is a strong function of its mass.

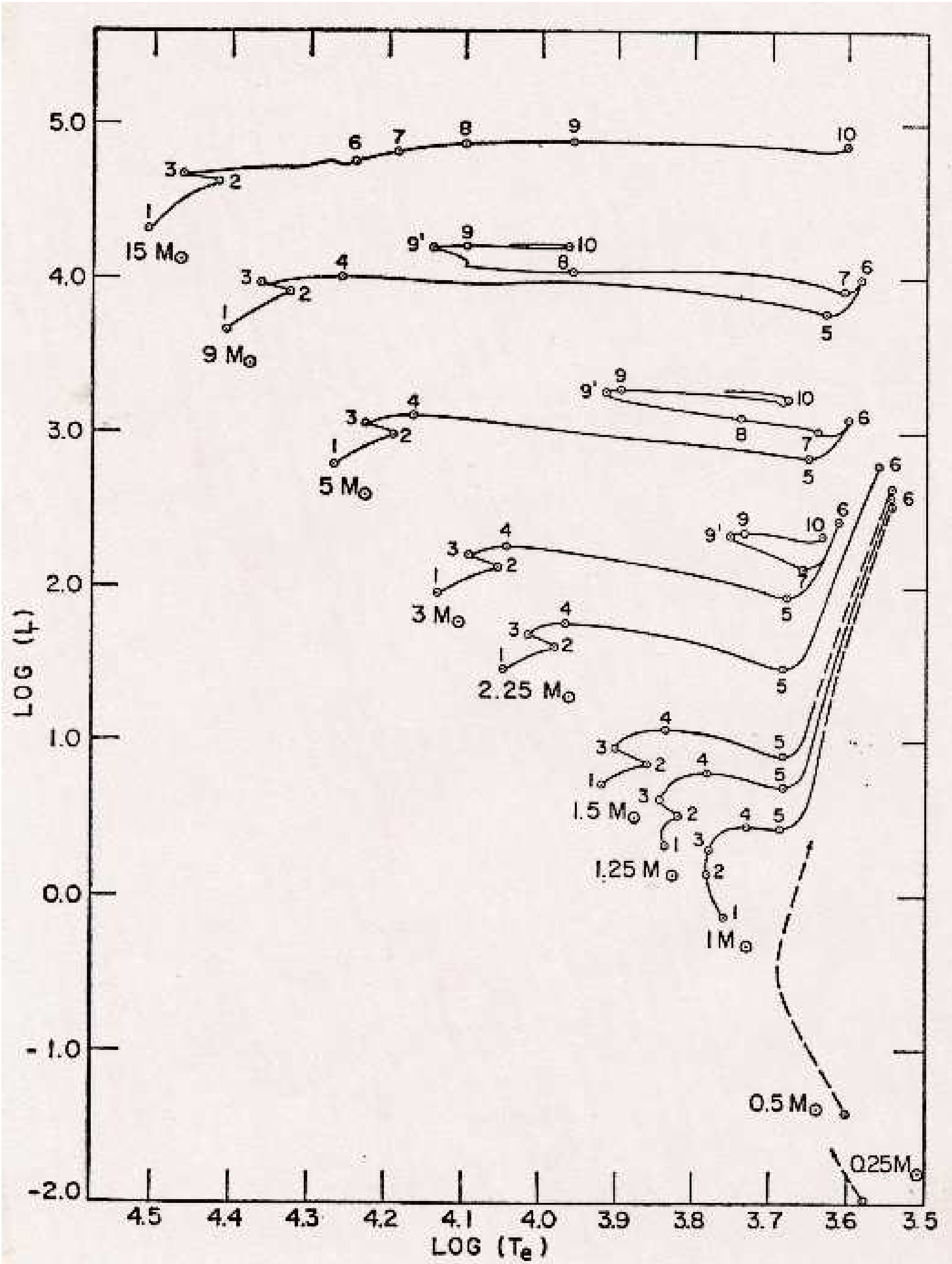
When the mass fraction of hydrogen in a stellar core declines to $X \sim 0.05$ (point 2 on the evolutionary track), the main-sequence phase has ended, and the star begins to undergo rapid changes.

• First, the entire star begins to contract. The core energy generation for this stage remains approximately constant, but the star increases its luminosity due to the conversion of gravitational energy to thermal energy. Simultaneously, the smaller stellar radius translates into a hotter effective temperature. For higher mass stars, the mass fraction of the convective core begins to shrink rapidly. On the diagram, the star travels from point 2 to 3. At point 3, the mass fraction of hydrogen in the core is $\sim 1\%$.

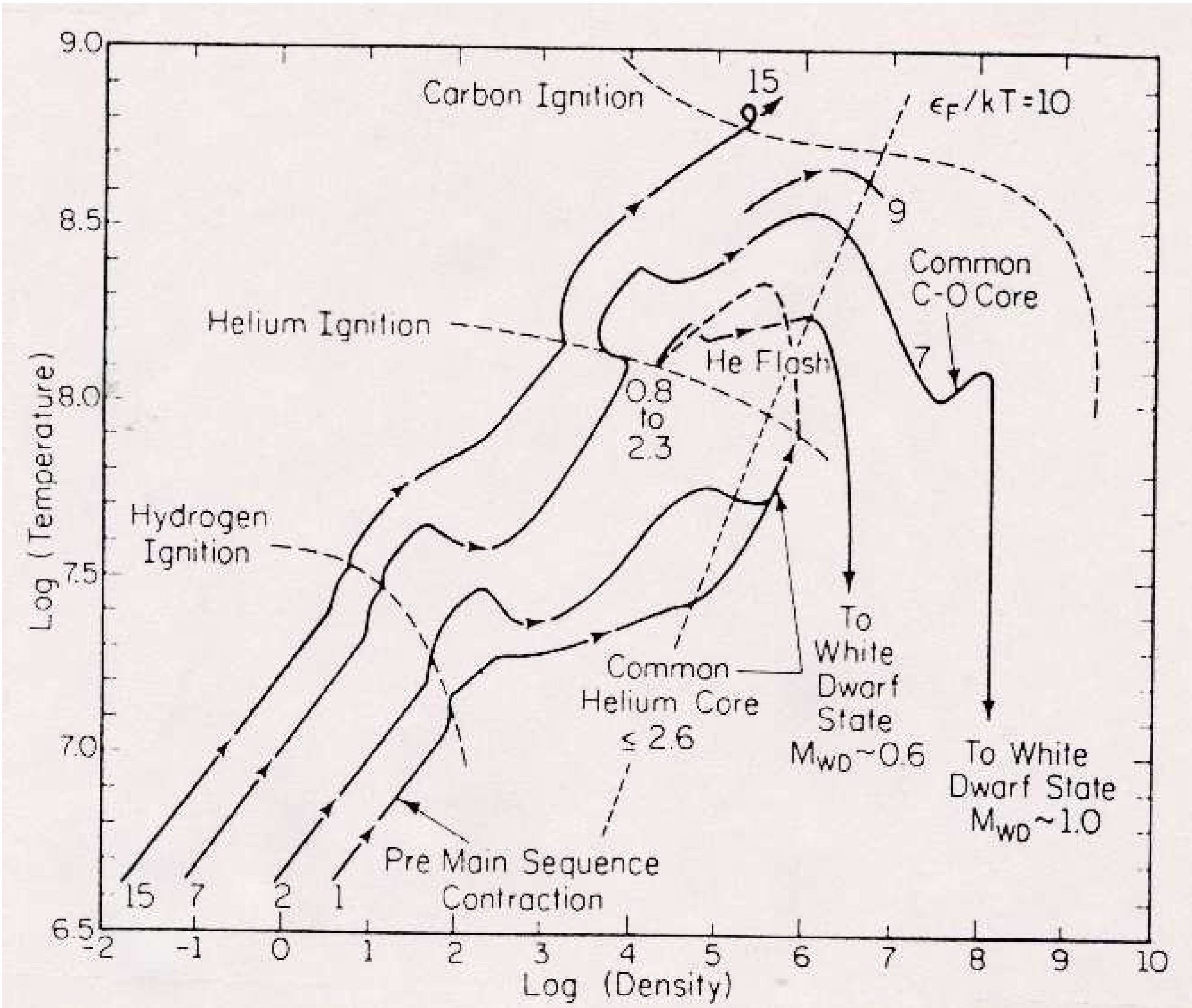
• When hydrogen fusion in a convective core stops, the core's leftover temperature gradient causes energy in the center to flow outward. Consequently, the core may temporarily cool, and begin to contract rapidly, turning some gravitational energy to thermal energy. The combination of these two effects makes the core isothermal. This condition slows the core's contraction, since the demand for energy flow from the center is decreased. (In stars with $\mathcal{M} > 10\mathcal{M}_\odot$, the core temperature will never actually decrease, while in stars with radiative cores, core contraction won't actually occur, since it has been adjusting to the decrease in hydrogen all along.)

• While this is happening, the hydrogen rich material around the core is drawn inward and eventually ignites in a thick shell, containing $\sim 5\%$ of the star's mass. Much of the energy from shell burning then goes into pushing matter away in both directions. As a result, the luminosity of the star does not increase; instead the outer part of the star expands. This "thick shell" phase continues with the shell moving outward in mass, until the core contains $\sim 10\%$ of the stellar mass (point 4). This is the Schönberg-Chandrasekhar limit. Stars with larger (by mass fraction) cores will reach this point faster than stars with small cores.)

• Light elements such as lithium, boron, and beryllium fuse at temperatures much lower than those of the central core. As a result, by the time a star has moved off the main sequence, these elements have been destroyed over the inner 98% of the star.



The evolution of stars in the HR diagram. For low-mass stars, the tracks only extend to the helium flash.



The evolution of the stellar cores. The density is given in g cm^{-3}.

The Schönberg-Chandrasekhar Limit

At first, a star evolving off the main sequence will roughly follow the predictions of homology. Specifically, its luminosity will increase due to the increase in the star's mean molecular weight, while its radius will remain roughly constant. This will continue until the hydrogen fraction $X < 0.05$. When the hydrogen runs out, the star's structure must change dramatically (non-homologously).

To see this, consider a star whose envelope is still behaving in an homologous fashion, but with a core that has no nuclear burning. If the core is not producing any luminosity, then to be in near thermal equilibrium, it must be approximately isothermal. (Otherwise, the energy will diffuse outward.) We can compute the maximum surface pressure an isothermal core of temperature T_c can withstand from hydrostatic equilibrium and the virial theorum. From hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)}{r^2}\rho$$

If we multiply this equation by $4\pi r^3 dr$, and integrate by parts over the core, the equation becomes

$$4\pi R^3 P\Big]_0^{R_c} - \int_0^{R_c} 12\pi r^2 P\, dr = -\int_0^{R_c} \frac{G\mathcal{M}\rho}{r^2} 4\pi r^3 dr$$

or

$$4\pi R_c^3\, P(R_c) - \int_0^{M_c} 3\,\frac{P}{\rho}\, d\mathcal{M} = -\int_0^{\mathcal{M}_c} \frac{G\mathcal{M}}{r} d\mathcal{M}$$

At the surface of the core, the equation of state is close to that of an ideal gas, so $P/\rho \propto T$. The pressure at the surface of the core can therefore be re-written as

$$P(R_c) = K_1 \frac{\mathcal{M}_c T_c}{R_c^3} - K_2 \frac{\mathcal{M}_c^2}{R_c^4} \tag{21.1.1}$$

The maximum surface pressure a core with size R_c and $\mathcal{M}_c$ can have can then be found by setting the derivative of $P(R_c)$ to zero, *i.e.*,

$$\frac{dP(R_c)}{dR_c} = -3K_1 \frac{\mathcal{M}_c T_c}{R_c^4} + 4K_2 \frac{\mathcal{M}_c^2}{R_c^5} = 0$$

or

$$R_c(\max) = K_3 \frac{\mathcal{M}_c}{T_c} \tag{21.1.2}$$

and

$$P(R_c) < K_4 \frac{T_c^4}{\mathcal{M}_c^2} \tag{21.1.3}$$

Meanwhile, we can independently compute the pressure at the core-envelope boundary under the assumption that the envelope physics is homologous. Recall from (17.4) that

$$P \propto \mathcal{M}^2 r^{-4} \tag{21.1.4}$$

Now if we assume an ideal gas equation of state ($\alpha = \delta = 1$), then from (17.13),

$$\log \frac{R}{R_0} = \frac{1}{2}(1 + C_1) \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}$$

$$\log \frac{T}{T_0} = \frac{1}{2\delta} \{1 + (3 - 4\alpha) C_1\} \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} = \frac{1}{2}(1 - C_1) \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}}$$

If we add these equations, we get

$$\log \frac{T}{T_0} + \log \frac{R}{R_0} = \frac{1}{2} \log \frac{\mathcal{M}_T}{\mathcal{M}_{T_0}} \{1 + C_1 + 1 - C_1\}$$

which gives

$$T_c \propto \frac{\mathcal{M}_T}{r} \tag{21.1.5}$$

Combining this with (21.1.4), we get

$$P \propto K_5 \frac{T_c^4}{\mathcal{M}_T^2} \tag{21.1.6}$$

Thus, the pressure at the surface of the isothermal core is independent of the size of the core (R_c), but is inversely proportional to $\mathcal{M}_c^2$ (from the hydrostatic equilibrium) and $\mathcal{M}_T^2$ (from homology). Setting (21.1.3) and (21.1.6) equal, we get

$$\frac{\mathcal{M}_C}{\mathcal{M}_T} < \left(\frac{K_4}{K_5} \right)^{1/2} = q_{\text{sc}}$$

In other words, under the assumption of virial equilibrium and homology, there's a maximum limit to the mass fraction of an isothermal core. Once $\mathcal{M}_c/\mathcal{M}_T \sim 0.1$, no stable configuration exists. The star must adopt a different equilibrium.

The Hertzsprung Gap and the Subgiant Phase

• When the Schönberg-Chandrasekhar limit is reached, the star must change its structure. First, core contraction begins to occur on the Kelvin-Helmholtz timescale, and the rapid increase in core density causes an increase in the temperatures and densities in the shell surrounding the core. The result is a strong increase in the rate of nuclear reactions in the shell, which again pushes matter away on both sides. The result is a very thin zone of hydrogen burning; as the star evolves from point 4 to point 5, the mass of the shell will decrease from $\sim 3\%$ to $\sim 0.5\%$ the mass of the star.

• Although the reaction rates in the shell increase as the shell narrows (due to the strong temperature dependence of nuclear reaction rates), this is compensated for by the decrease in the shell mass. As a result, the total luminosity produced by the shell decreases slightly. Moreover, much of this energy goes into mechanical work; because the specific heat of the star is negative, the increased temperature causes the envelope to expand and cool. About half the energy needed to expand the envelope comes from shell burning; the other half comes from the envelope itself, as it adjusts to the new conditions. The envelope cooling rapidly moves the star to the right in the HR diagram, across the "Hertzsprung Gap," until it reaches the limiting Hayashi line.

• The effective temperature where the star begins to ascend the giant branch is approximately independent of mass. It is, however, very sensitive to metallicity, since the electrons provided by metals are providing most of the surface opacity via H^- absorption. Note also that because the convective envelope is large, an uncertainty in the mixing length translates into a significant displacement of position of the giant branch, with

$$\delta \log T_{\text{eff}} \sim \left[0.02 \log \frac{\mathcal{L}}{\mathcal{L}_\odot} + 0.143 \right] \delta \log \alpha \approx 0.16 \log \alpha$$

• While in the shell burning phase, the rapidly increasing central densities and temperatures can temporarily provide an additional source of nuclear energy for the star. While the core of the star is mostly helium, there are some trace amounts of metals, in particular CNO. Because CNO burning has gone to equilibrium, most of the CNO nuclei will be in the form of ^{14}N. As the core contracts and the conditions becomes more extreme, ^{14}N can fuse with helium, via the reaction $^{14}\mathrm{N}(^{4}\mathrm{He}, \gamma)^{18}\mathrm{F}(\beta^{+}, \nu)^{18}\mathrm{O}$. This rapidly changes all the ^{14}N in the star to ^{18}O, and gives the star a little extra life. The place where this occurs depends on the mass of the star. Stars with $\mathcal{M} > 9\mathcal{M}_\odot$ fuse ^{14}N in the Hertzsprung Gap; as a result, they increase their luminosity on their trip across the HR diagram. Stars with $\mathcal{M} \sim 5\mathcal{M}_\odot$ burn ^{14}N on the giant branch. For stars with $\mathcal{M} < 3\mathcal{M}_\odot$, ^{14}N burning on the giant branch actually causes them to temporarily *descend* the branch, as the core-shell region adjusts to the new energy. None of these effects is directly observable, but ^{14}N will increase the lifetime of a subgiant star slightly.

The Giant Branch

On the giant branch, the size of the hydrogen-burning shell continues to decrease. However, unlike the thick-burning shell phase, the decrease in mass is more than offset by the higher nuclear reaction rates. To appreciate this, consider that the *luminosity produced by shell burning depends almost exclusively on the mass of the core.* The reason for this comes from the equation of hydrostatic equilibrium: at the surface of a compact, highly condensed core

$$\frac{dP}{d\mathcal{M}} = -\frac{G\mathcal{M}}{4\pi r^4} \ll 0$$

Thus the pressure drops by many orders of magnitude in just a small region. The extended envelope above the shell is virtually weightless, and has almost no influence on the properties of the shell.

The dependence of luminosity on core mass can be estimated from homology. If we assume the density, temperature, pressure, and luminosity of a burning shell go as some power of the mass and radius of the core, *i.e.,*

$$\rho \propto \mathcal{M}_c^{\phi_1} R_c^{\phi_2}; \quad T \propto \mathcal{M}_c^{\psi_1} R_c^{\psi_2}; \quad P \propto \mathcal{M}_c^{\tau_1} R_c^{\tau_2}; \quad \mathcal{L} \propto \mathcal{M}_c^{\sigma_1} R_c^{\sigma_2} \tag{21.2.1}$$

and use homology on the ideal gas law (which is a good approximation for regions outside the core), then

$$\frac{P}{P_0} = \frac{\rho}{\rho_0}\frac{T}{T_0} \Longrightarrow \left(\frac{\mathcal{M}_c}{\mathcal{M}_c'}\right)^{\tau_1} \left(\frac{R_c}{R_c'}\right)^{\tau_2} = \left(\frac{\mathcal{M}_c}{\mathcal{M}_c'}\right)^{\phi_1+\psi_1} \left(\frac{R_c}{R_c'}\right)^{\phi_2+\psi_2}$$

giving

$$\tau_1 = \phi_1 + \psi_1 \qquad \text{and} \qquad \tau_2 = \phi_2 + \psi_2 \tag{21.2.2}$$

Similarly, the Eulerian equation of hydrostatic equilibrium yields

$$\left(\frac{dP}{dr}\right)\left(\frac{dr}{dr'}\right)\left(\frac{dP'}{dP}\right) = -\frac{G\mathcal{M}_c}{R_c^2}\rho\left(\frac{R_c}{R'_c}\right)\left(\frac{dP'}{dP}\right) = -\frac{G\mathcal{M}'_c}{{R'_c}^2}\rho'$$

which gives

$$dP = \left(\frac{\mathcal{M}_c}{\mathcal{M}'_c}\right)\left(\frac{R'_c}{R_c}\right)\left(\frac{\rho}{\rho'}\right) = dP'$$

When integrated to a region outside the shell (with negligible pressure), this equation and (21.2.2) imply

$$\tau_1 = \phi_1 + 1 \qquad \text{and} \qquad \tau_2 = \phi_2 - 1 \tag{21.2.3}$$

In Eulerian coordinates, the luminosity equation is

$$\left(\frac{d\mathcal{L}}{dr}\right)\left(\frac{dr}{dr'}\right)\left(\frac{d\mathcal{L}'}{d\mathcal{L}}\right) = 4\pi R_c^2\rho\,\epsilon\left(\frac{R_c}{R'_c}\right)\left(\frac{d\mathcal{L}'}{d\mathcal{L}}\right) = 4\pi {R'_c}^2\rho'\epsilon'$$

If we substitute using $\epsilon = \epsilon_0\rho^\lambda T^\nu$, then

$$d\mathcal{L} = \left(\frac{R_c}{R'_c}\right)^3\left(\frac{\rho}{\rho'}\right)^{\lambda+1}\left(\frac{T}{T'}\right)^\nu d\mathcal{L}'$$

Again, we can integrate over the shell until energy generation vanishes, and re-write the density and temperature in terms of the core mass and radius to get

$$\sigma_1 = (\lambda+1)\phi_1 + \psi_1\nu \qquad \text{and} \qquad \sigma_2 = (\lambda+1)\phi_2 + \psi_2\nu + 3 \tag{21.2.4}$$

Finally, we can write the equation for the radiative energy transport that will take place in the region immediately adjacent to the shell.

$$\left(\frac{dT}{dr}\right)\left(\frac{dr}{dr'}\right)\left(\frac{dT'}{dT}\right) = -\left(\frac{3\kappa\rho\mathcal{L}}{16\pi acR_c^2T^3}\right)\left(\frac{R_c}{R_c'}\right)\left(\frac{dT'}{dT}\right) = -\frac{3\kappa'\rho'\mathcal{L}'}{16\pi acR_c'^2T'^3}$$

Substituting for κ using $\kappa = \kappa_0 P^s T^t$, we get

$$\left(\frac{\mathcal{L}}{\mathcal{L}'}\right)\left(\frac{R_c}{R_c'}\right)^{-1}\left(\frac{\rho}{\rho'}\right)\left(\frac{P}{P'}\right)^s\left(\frac{T}{T'}\right)^{t-3} = \frac{dT}{dT'}$$

This can once again be integrated to a region with negligible temperature, and re-written in terms of the core parameters. The result is

$$(4-t)\psi_1 = \sigma_1 + \phi_1 + s\tau_1 \qquad \text{and} \qquad (4-t)\psi_2 = \sigma_2 + \phi_2 - 1 + s\tau_2 \tag{21.2.5}$$

Equations (21.2.2) - (21.2.5) form a set of 8 (non-linear) equations with 8 unknowns. With (a great deal of) effort, these can be solved to give

$$\phi_1 = -\frac{\nu - 4 + s + t}{s + \lambda + 2} \qquad \phi_2 = \frac{\nu - 6 + s + t}{s + \lambda + 2}$$

$$\psi_1 = 1 \qquad \psi_2 = -1$$

$$\tau_1 = 1 + \phi_1 \qquad \tau_2 = \phi_2 - 1$$

$$\sigma_1 = \frac{(4 - s - t)(\lambda + 1) + \nu(s + 1)}{s + \lambda + 2}$$

$$\sigma_2 = \frac{(s + t)(\lambda + 1) + 3(s - \lambda) - \nu(s + 1)}{s + \lambda + 2} \tag{21.2.6}$$

Note how the luminosity of a shell burning star behaves. For electron scattering ($s = t = 0$) and CNO burning ($\lambda = 1$, $\nu = 13$), $\mathcal{L} \propto \mathcal{M}_c^7 R_c^{-16/3}$; for electron scattering and helium burning ($\lambda = 2$, $\nu = 40$), $\mathcal{L} \propto \mathcal{M}_c^{13} R_c^{-23/2}$. Unlike core burning, the luminosity of a shell burning star depends both on the core mass and the energy generation mechanism. Note also that the temperature of the shell has a very simple relation reflective of the virial theorum: $T \propto \mathcal{M}_c / R_c$.

We can make further progress by relating the mass of the stellar core to its radius. If the core were fully degenerate, this would be easy: we could use the mass-radius relation for white dwarfs. However, the region immediately below the shell is not degenerate (due to the high temperature of the shell), and, although this region can have a negligibly small mass, it can occupy a substantial fraction of the core's volume. Thus, a better approximation for the core mass-core radius relation of a shell burning star is a two zone model consisting of a degenerate core (obeying a polytropic mass-radius relation), and an isothermal, ideal gas transition region.

For the transition region, we can combine the ideal gas law with hydrostatic equilibrium to get

$$\frac{kT}{\mu m_a} \frac{d\rho}{dr} = -\frac{G\mathcal{M}}{r^2}\rho \Longrightarrow \; d\ln\rho = -\frac{G\mu m_a}{k}\frac{\mathcal{M}}{T}\frac{1}{r^2}dr$$

This can be integrated from the top of the core, R_c, to the place where electron degeneracy takes over, r_t. The result is

$$\ln\rho_t - \ln\rho_c = -\frac{G\mu m_a}{k}\left(\frac{\mathcal{M}_c}{TR_c}\right)\left\{\frac{R_c}{r_t} - 1\right\} \qquad (21.2.7)$$

This equation can quickly be simplified. First, since the transition region is at the same temperature as the shell, then from (21.2.6),

$T \propto \mathcal{M}_c/R_c$, and the term in parenthesis is a constant (C_1). Next, we can estimate ρ_t by equating the ideal gas and electron degeneracy density laws. From (7.3.8), we have

$$\ln \rho_t = \frac{3}{2} \ln T + \frac{5}{2} \ln \mu_e - \frac{3}{2} \ln \mu - 17.55 = \frac{3}{2} \ln T + C_2 \quad (21.2.8)$$

Similarly, we can substitute for r_t using the polytropic mass-radius relation for white dwarfs

$$r_t = (4\pi)^{\frac{1}{n-3}} \left[\frac{(n+1)K}{G}\right]^{\frac{n}{3-n}} \left[-\xi'^2 \frac{d\theta}{d\xi}\right]^{\frac{n-1}{3-n}} \mathcal{M}_c^{\frac{1-n}{3-n}}$$

$$= 9.71 \times 10^{19} \mu_e^{-5/3} \mathcal{M}_c^{-1/3} = C_4 \mu_e^{-5/3} \mathcal{M}_c^{-1/3} \quad (16.1.4)$$

and for $\ln \rho_c$ using the homology relations for (21.2.6)

$$\ln \rho_c = C_3 + \phi_1 \ln \mathcal{M}_c + \phi_2 \ln R_c$$

This gives us a relation between R_c, T, μ, μ_e, and $\mathcal{M}_c$

$$\frac{3}{2} \ln T + C_2 - C_3 - \phi_1 \ln \mathcal{M}_c - \phi_2 \ln R_c = -\frac{G \mu m_a}{k\, C_1} \left(\frac{R_c \mathcal{M}_c^{1/3}}{C_4 \mu_e^{-5/3}} - 1 \right) \quad (21.2.9)$$

For reasonable values, this equation yields

$$\frac{d \ln R_c}{d \ln \mathcal{M}_c} \sim -0.16 \quad (21.2.10)$$

(a rather weak dependence). The net result is that

$$\mathcal{L} \propto \mathcal{M}_c^z \quad (21.2.11)$$

with $z \sim 8$ for CNO burning shells, and $z \sim 15$ for helium burning shells.

Once we have an equation for the luminosity of a shell burning star as a function of core-mass, it is then relatively easy to calculate the luminosity of the star as a function of time. Consider that as a shell burning star evolves, it continually deposits more mass on its central core, with

$$\frac{d\mathcal{M}_c}{dt} = \frac{\mathcal{L}}{XQ}$$

where X is the mass fraction of the fuel, and Q the amount of energy generated by one gram of material. Thus

$$\int_0^{t_1} dt = \frac{XQ}{K} \int_{\mathcal{M}_{C_0}}^{\mathcal{M}_{C_1}} \mathcal{M}^{-z} d\mathcal{M} \tag{21.2.12}$$

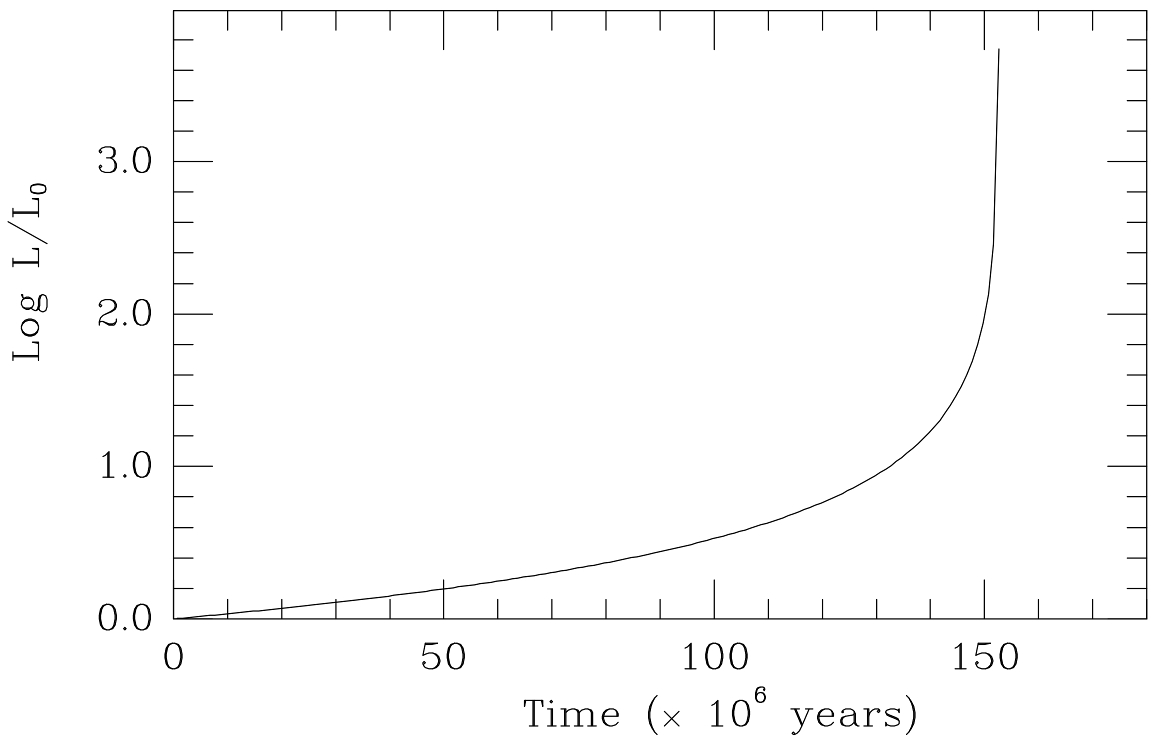
where $t = 0$ is the fiducial time when $\mathcal{L} = \mathcal{L}_0$, and K is the constant of proportionality relating $\mathcal{L}$ to $\mathcal{M}_c$. Equation (21.2.12) is easily integrated to yield

$$\frac{QX}{(1-z)K} \left\{ \mathcal{M}_{C_1}^{1-z} - \mathcal{M}_{C_0}^{1-z} \right\} = t$$

which, after a bit of manipulation, becomes

$$\mathcal{L}(t) = \left[\frac{(1-z)K^{1/z}}{QX\mathcal{L}_0^{(1-z)/z}} \, t + 1 \right]^{z/(1-z)} \tag{21.2.13}$$

This is a sharply increasing exponential! The star's evolution accelerates dramatically as it ascends the giant branch.



Log L/L_0
3.0
2.0
1.0
0.0
0
50
100
150
Time ($\times 10^6$ years)