White Dwarf Stars

After nuclear burning ceases, a post-AGB star rapidly becomes a white dwarf. Although gravitational contraction will provide some luminosity for a while, the luminosity evolution of the star can be well modeled as simple cooling for a highly conductive isothermal, degenerate core blanketed by a radiative non-degenerate envelope.

A simple way to model the cooling of a white dwarf is to use a two-zone model consisting of a degenerate, non-relativistic, isothermal core covered by a thin layer of ideal gas. From (7.3.8), the density at the transition region will be

$$\rho_t = \left(\frac{20 m_e k}{\mu}\right)^{3/2} \left(\frac{\pi}{3 N_A h^3}\right) \mu_e^{5/2}\, T^{3/2} \;=\; C_0\, \mu_e^{5/2}\, \mu^{-3/2}\, T_c^{3/2} \tag{25.1.1}$$

where T_c is the core temperature. From the ideal gas law, the pressure at this location will be

$$P_t = \frac{N_A \rho_t}{\mu} k\, T_c = C_0 \left(\frac{\mu_e}{\mu}\right)^{5/2} N_A k\, T_c^{5/2} \tag{25.1.2}$$

Now consider the behavior of a thin radiative atmosphere, which is neither a source nor sink of luminosity. Since this layer is thin, the mass of the atmosphere is negligible. Hence, if we adopt an opacity law of the form

$$\kappa = \kappa_0 \rho^s T^t = \kappa_0 \left(\frac{\mu}{N_A k}\right)^s P^s\, T^{t-s}$$

then from (3.1.6)

$$\nabla_{\rm rad} = \frac{P}{T}\frac{dT}{dP} = \frac{3\kappa}{16\pi ac\,G}\frac{\mathcal{L}}{\mathcal{M}_T}\,\frac{P}{T^4}$$

$$= \frac{3\kappa_0}{16\pi ac\,G}\left(\frac{\mu}{N_A k}\right)^s \frac{\mathcal{L}}{\mathcal{M}_T} P^{s+1}\,T^{\,t-s-4}$$

$$= C_1\,\mu^s\,\left(\frac{\mathcal{L}}{\mathcal{M}_T}\right) P^{s+1}\,T^{\,t-s-4} \tag{25.1.3}$$

Following our analysis of the radiative atmospheres of normal stars, we can integrate this expression from the stellar photosphere down to the transition layer and obtain

$$T_c^{4+s-t}\left\{1 - \left(\frac{T_p}{T_c}\right)^{4+s-t}\right\} =$$

$$\frac{4+s-t}{1+s}\,C_1\,\mu^s\,\left(\frac{\mathcal{L}}{\mathcal{M}_T}\right) P_t^{1+s}\left\{1 - \left(\frac{P_p}{P_t}\right)^{1+s}\right\} \tag{9.15}$$

Since the pressure and temperature at the transition region will be much larger than at the surface, the terms in the parentheses vanish, and

$$T_c^{4+s-t} \;=\; \frac{4+s-t}{1+s}\;C_1\;\mu^s\;\left(\frac{\mathcal{L}}{\mathcal{M}_T}\right) P_t^{1+s}$$

or

$$P_t \;=\; \left\{\frac{\mathcal{M}_T\,(1+s)}{\mathcal{L}\,C_1\,\mu^s\,(4+s-t)}\right\}^{1/1+s}\;T_c^{(4+s-t)/(1+s)} \tag{25.1.4}$$

If we equate this equation to (25.1.2), we get an expression for the luminosity and temperature of the star in terms of the temperature of the central core

$$C_0 \left(\frac{\mu_e}{\mu}\right)^{\frac{5}{2}} N_A k \, T_c^{\frac{5}{2}} = \left\{ \frac{\mathcal{M}_T \, (1+s)}{\mathcal{L} C_1 \, \mu^s \, (4+s-t)} \right\}^{\frac{1}{1+s}} T_c^{\frac{(4+s-t)}{(1+s)}} \implies$$

$$\mathcal{L} = \left\{ \frac{1+s}{C_1 \, (4+s-t)} \right\} \left\{ \frac{\mu_e^{-5/2}}{C_0 N_A k} \right\}^{1+s} \mu^{(3s+5)/2} \, \mathcal{M}_T \, T_c^{(3-3s-2t)/2} \tag{25.1.5}$$

For a Kramers opacity which is dominated by bound-free absorption, $s = 1$, $t = -7/2$, and $\kappa_0 \approx 4 \times 10^{25}$ cm^2-g^{-1}. Moreover, by (5.1.3), (5.1.6), and (5.1.7), $\mu_e \approx 2$, and $\mu \approx 1.75$, so

$$\mathcal{L} = C_2 \, \mathcal{M}_T \, T_c^{7/2} \tag{25.1.6}$$

(where $C_2 \sim 5 \times 10^{-30}$, if the mass and luminosity are in solar units).

Next, consider the reaction of the core to its energy loss. The core is already degenerate, so gravitational contraction will not occur. However the core will cool, and the amount of this cooling will be given by the specific heat.

$$\mathcal{L} = -\frac{dE}{dt} = -c_V \mathcal{M} \frac{dT_c}{dt} \tag{25.1.7}$$

From this, we can compute the cooling curve of the star. If we take the derivative of (25.1.6) with respect to time, write it in terms of luminosity and mass, and then substitute in for the temperature derivative using (25.1.7), then

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \frac{7}{2} C_2\, \mathcal{M}_T\, T_c^{5/2}\, \frac{dT_c}{dt} \\
&= \frac{7}{2} C_2\, \mathcal{M}_T \left(\frac{\mathcal{L}}{C_2 \mathcal{M}_T} \right)^{5/7} \frac{dT_c}{dt} \\
&= \frac{7}{2} C_2\, \mathcal{M}_T \left(\frac{\mathcal{L}}{C_2 \mathcal{M}_T} \right)^{5/7} \left(-\frac{\mathcal{L}}{c_V \mathcal{M}_T} \right) \\
\frac{d\mathcal{L}}{dt} &= -\frac{7}{2c_V}\, C_2^{2/7} \mathcal{M}_T^{-5/7} \mathcal{L}^{12/7} \qquad (25.1.8)
\end{aligned}
$$

or

$$
\mathcal{L}^{-12/7} d\mathcal{L} = -\frac{7}{2c_V}\, C_2^{2/7} \mathcal{M}_T^{-5/7} dt
$$

This can be integrated easily to yield

$$
t_{\text{cool}} = \frac{2}{5} C_2^{-2/7} c_V \mathcal{M}_T^{5/7} \left\{ \mathcal{L}^{-5/7} - \mathcal{L}_0^{-5/7} \right\} \qquad (25.1.9)
$$

To calculate the specific heat, we can take advantage of the fact that under degenerate conditions, the specific heat of electrons is negligible (see Chandrasekhar, *Stellar Structure,* if you really want to know the details). Thus c_V is given almost entirely by the ions

$$
c_V = \left(\frac{dE}{dT} \right)_V = \frac{d}{dT} \left\{ \frac{3}{2} \frac{N_A k\, T}{\mu_I} \right\} = \frac{3}{2} \frac{N_A\, k}{\mu_I} \qquad (25.1.10)
$$

If we plug in the numbers, then in solar units

$$
t_{\text{cool}} = 2 \times 10^8\, \mu_I^{-1}\, \mathcal{M}_T^{5/7} \left\{ \mathcal{L}^{-5/7} - \mathcal{L}_0^{-5/7} \right\} \text{ years} \qquad (25.1.11)
$$

Note that $\mu_I \sim 12$ if the white dwarf is entirely carbon.

Finally, to locate the star in the HR diagram we can again draw on our knowledge of radiative envelopes. The temperature structure of an atmosphere with negligible mass is given by

$$T_c - T_p = \left(\frac{\mu}{N_A\,k}\right)\left(\frac{1+s}{4+s-t}\right) G\mathcal{M}_T \left(\frac{1}{R_c} - \frac{1}{R_p}\right) \tag{9.18}$$

In this equation, the core temperature, T_c, is given as a function of luminosity by (25.1.6), the core radius comes from the polytropic relation between mass and radius (16.1.4), and the photospheric radius is related to the star's luminosity and photospheric temperature by the blackbody equation. Thus

$$T_c - T_p = A - B\,T_p^2 \tag{25.1.12}$$

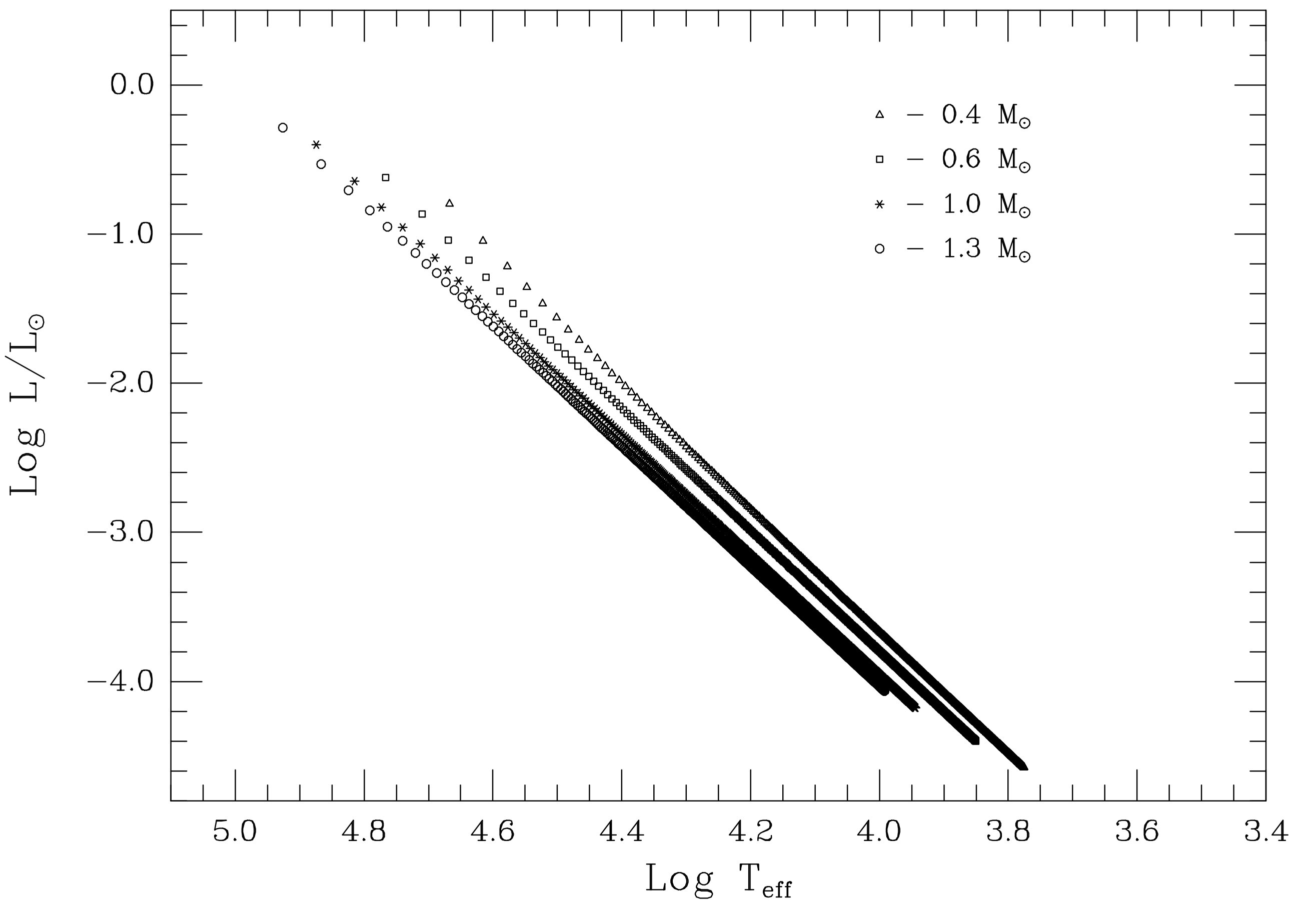
where

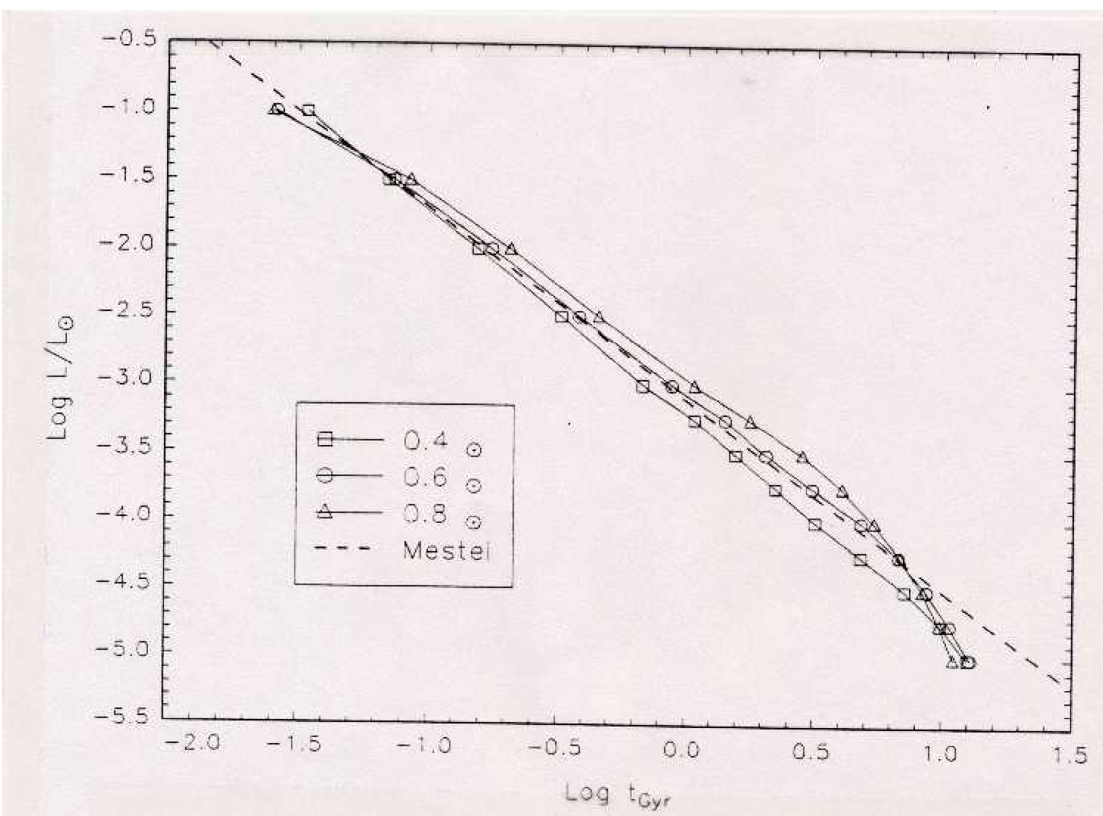
$$A = \left(\frac{\mu}{N_A\,k}\right)\left(\frac{1+s}{4+s-t}\right)\frac{G\mathcal{M}_T}{R_c}$$

and

$$B = \left(\frac{\mu}{N_A\,k}\right)\left(\frac{1+s}{4+s-t}\right) G\mathcal{M}_T \left(\frac{4\pi\sigma}{\mathcal{L}}\right)^{1/2}$$

Equations (25.1.12) is quadratic, but since the second term in the discriminant is much greater than 1, it essentially reduces to a linear function.


0.4 M⊙
0.6 M⊙
1.0 M⊙
1.3 M⊙
Log L/L⊙
Log Teff


0.4 ⊙
0.6 ⊙
0.8 ⊙
Mestel
Log L/L⊙
Log tGyr

This simple cooling law reproduces the observed distribution of white dwarfs quite well. The cooling timescale derived above is a factor of ~ 2 too fast (which probably comes from our definition of the transition region). However, the behavior of the cooling curve, and its position in the HR diagram is accurate. There are several features to note:

• The starting point for the calculation was $\mathcal{L} = 1000\mathcal{L}_{\odot}$ at $t = 0$, but this makes very little difference to the calculation. (This can be seen from (25.1.9) quite easily.) Thus the term $\mathcal{L}_0$ is usually dropped from cooling formulae.

• Each tick mark in the figure represents 10^7 years. Note that white dwarfs fade quickly at first; but after a while, their evolution is exceedingly slow. The last point plotted is after 10^{10} years; thus the coolest white dwarfs in existence should still have a temperature of $\gtrsim 6500$ K.

• The cooling curves for different mass stars are offset slightly. By matching the position of a white dwarf (or an evolved planetary nebula central star) with its position in the HR diagram, it is possible to estimate its mass.

• From a white dwarf's location in the HR diagram, it is possible to estimate how long it has been cooling, *i.e.*, its age. By examining the luminosity function of white dwarfs in the Milky Way, it is possible to get an independent estimate of the age of the Galaxy.

Realistic Models of White Dwarfs

There are a number of processes associated with white dwarfs that are difficult to model. As a result, our understanding of their cooling is not as complete as we would like.

• The composition of white dwarfs is not well known. Most are clearly a mixture of carbon and oxygen, but the proportion of these two elements is not well constrained. A few massive white dwarfs in nova systems show evidence of heavy elements; neon-oxygen-magnesium novae are relatively common.

• The surface layers of white dwarfs vary greatly. These differences are reflected in their spectral types

Type	Characteristics
DA	Balmer Lines only; no He I or metals present
DB	He I lines only; no H or metals present
DC	No lines of any type present
DO	He II present (extremely hot)
DZ	Only metal lines; no H or He present
DQ	Carbon lines present

Presumably, some of these differences depend on the details of the star's AGB evolution. However, it is still uncertain whether a DB white dwarf is born with no hydrogen, or whether trace amounts of hydrogen exist which later float to the surface.

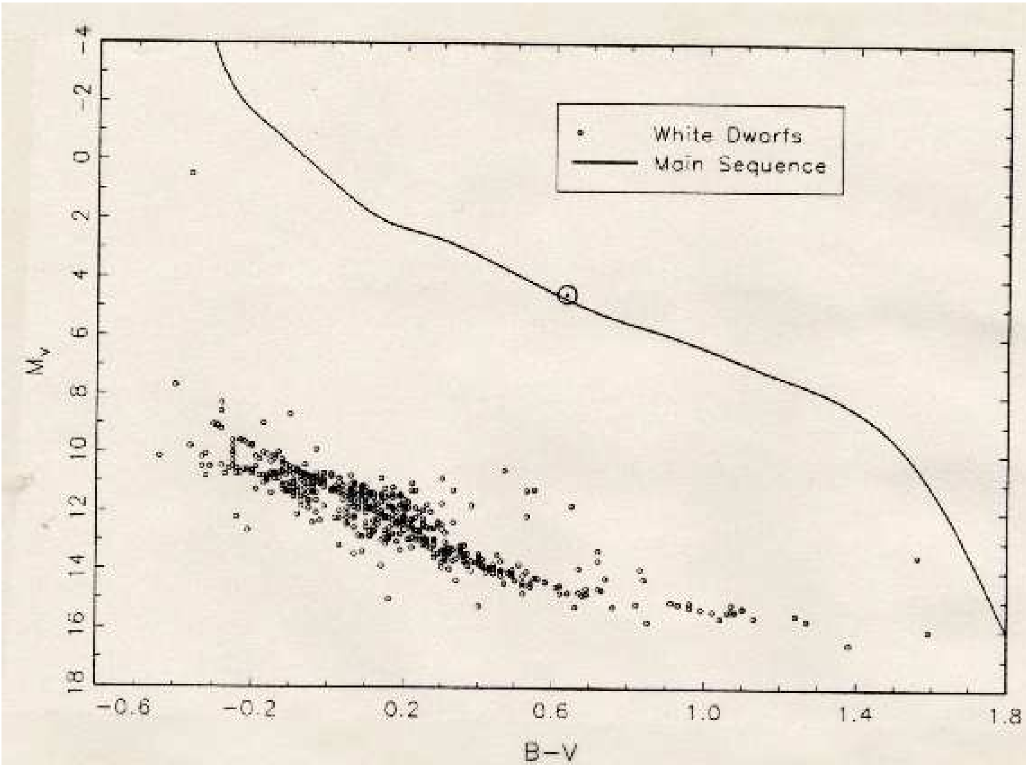
Note that this is not the only spectral classification scheme for white dwarfs. In particular, several schemes exist which connect the spectral features of white dwarfs to planetary nebulae nuclei.

Unfortunately, the classification criteria are not standard. (In fact, many are self-contradictory!)

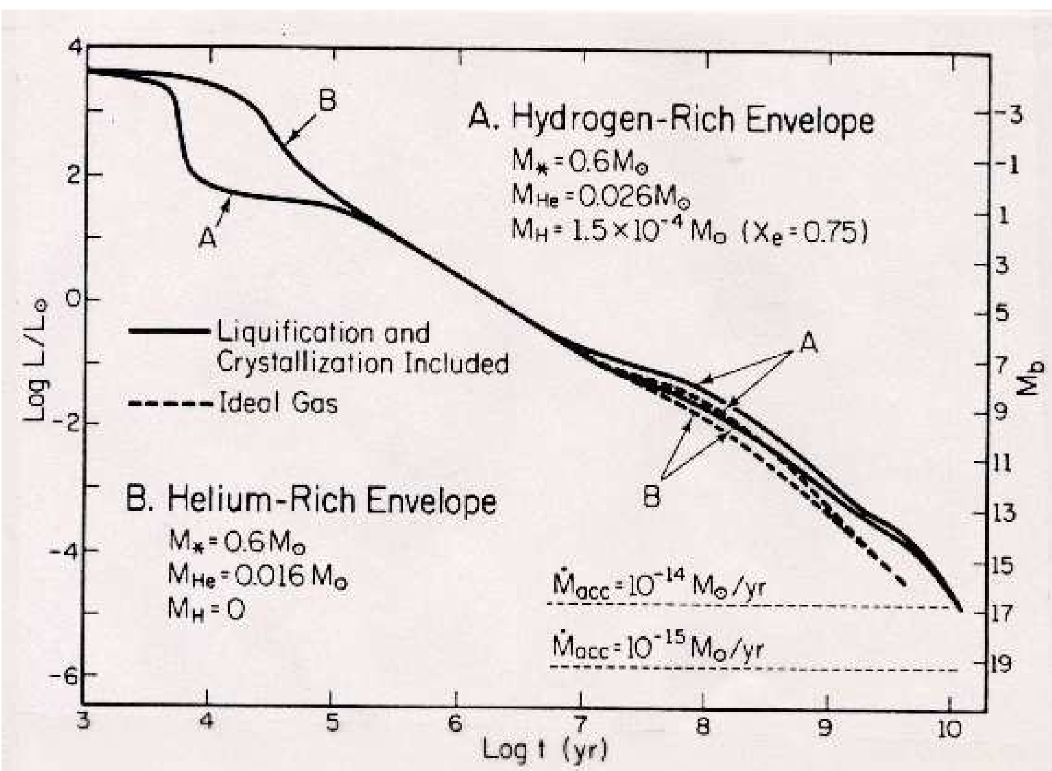
• As a white dwarf cools, solid-state effects becomes more important (*i.e.*, crystallization can occur). This greatly changes the equation of state.

• Many intermediate temperature white dwarf atmospheres do convect. Thus, simple radiative energy transport is not applicable to all white dwarfs. This changes the cooling curve somewhat, and can change the surface abundances.

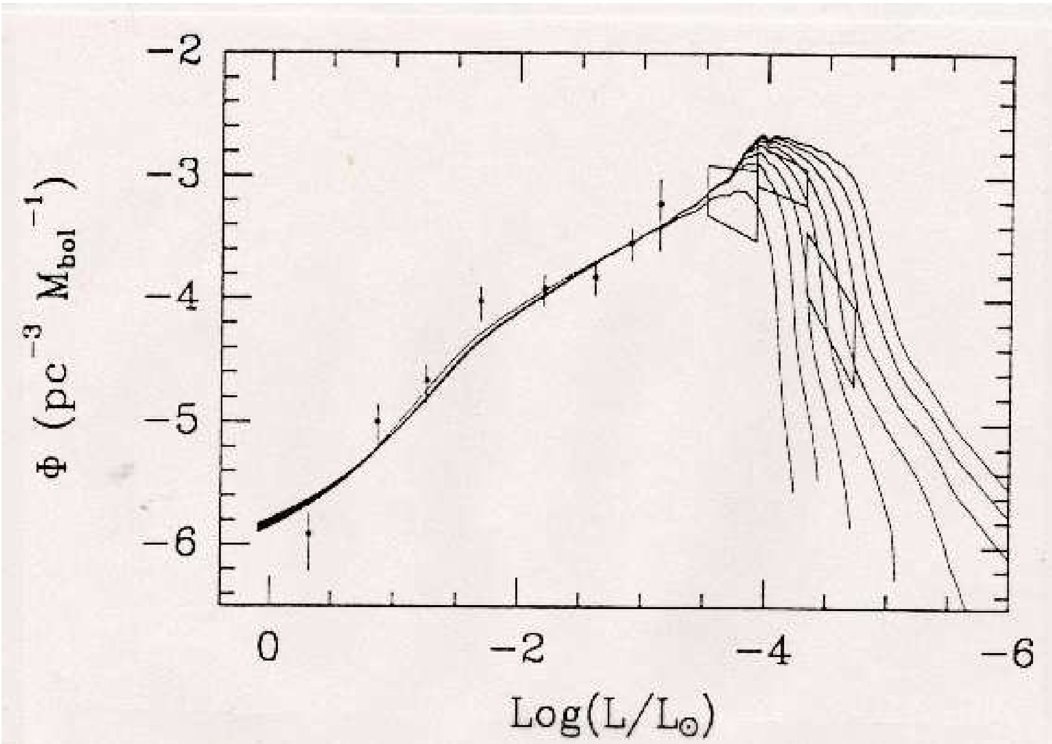
• It is an observed fact that the overwhelming majority of white dwarfs have masses near $\sim 0.6\mathcal{M}_\odot$, and there is very little dispersion about this mean. This is usually attributed to a very slowly varying initial-mass final-mass relation for stars. But the actual data is poor.



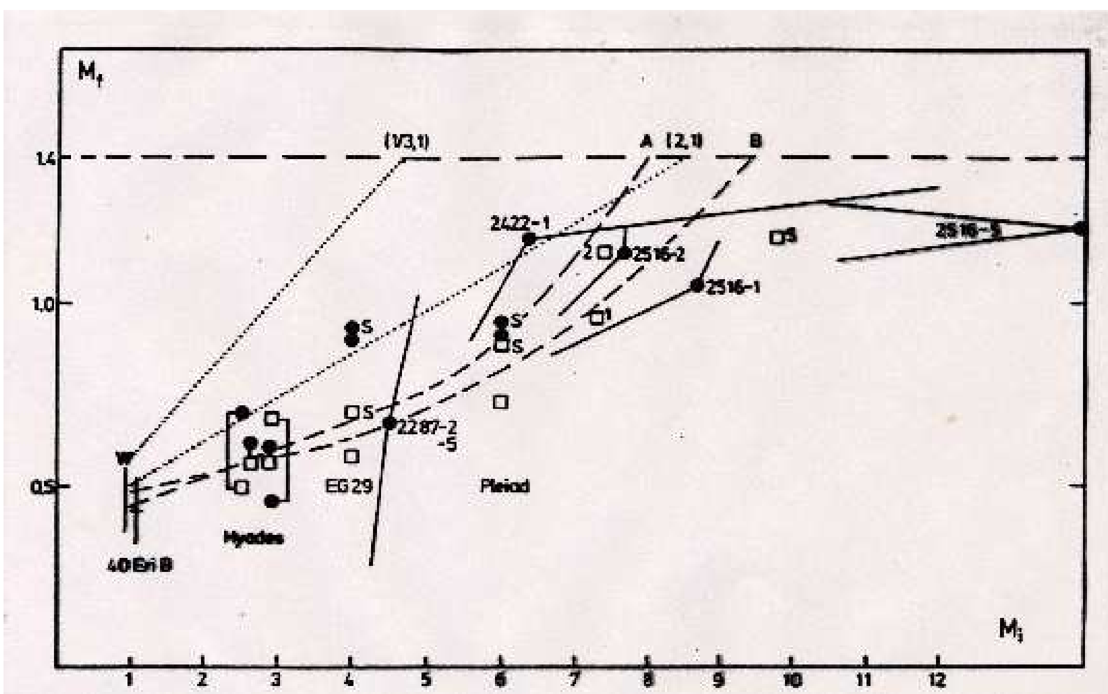
The locus of white dwarfs in the HR diagram.



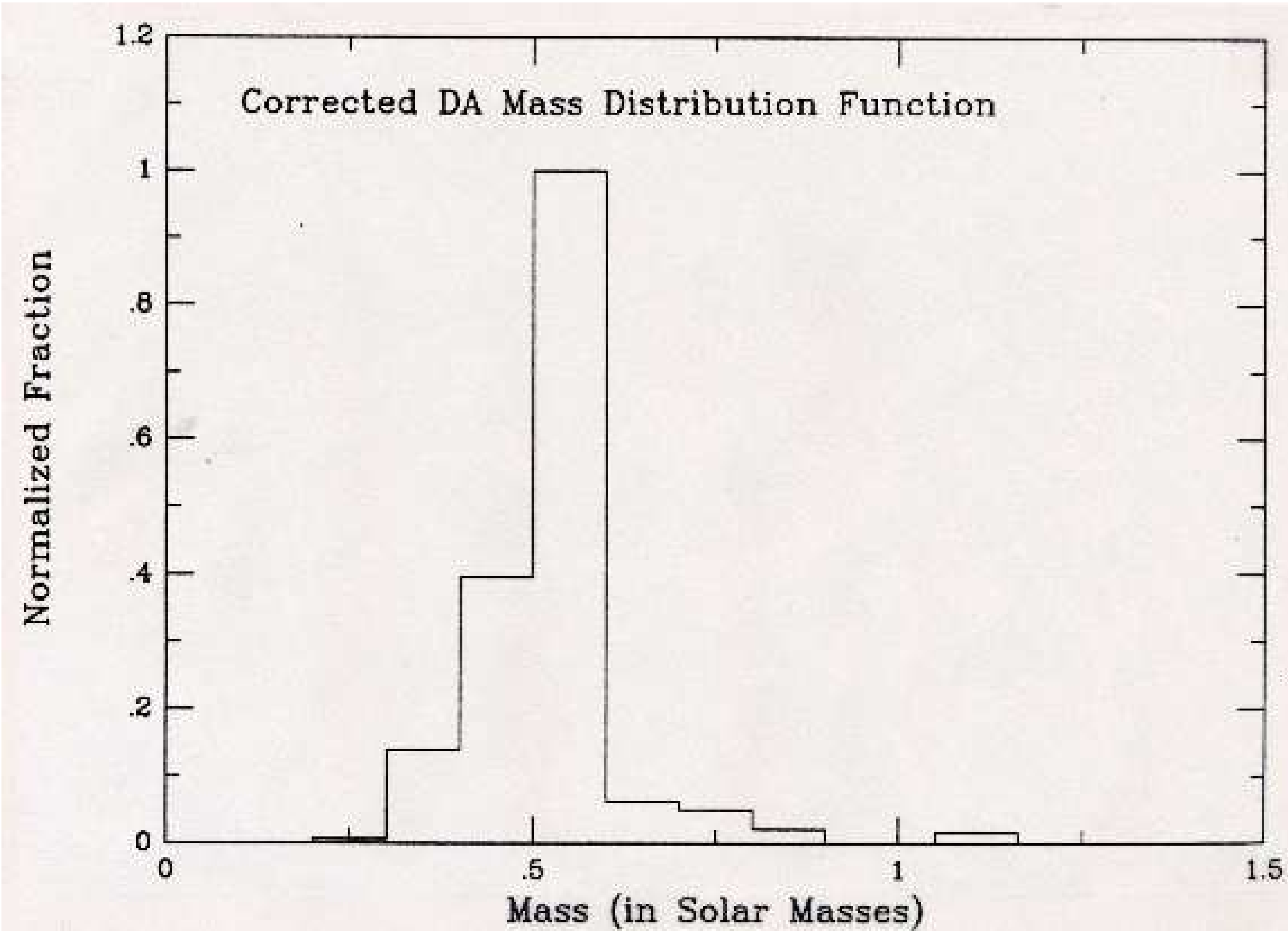
Cooling models for white dwarfs in the HR diagram.



The luminosity function of nearby white dwarfs.



The observed initial-mass final-mass relation (with error bars).



The mass distribution of nearby white dwarfs.

Pulsating Stars

Every star in the HR diagram has a natural (fundamental) frequency for pulsation. To understand this, consider a pulsation as the resonance of a sound wave. To first order, the speed at which a sound wave traverses a star is

$$v_s = \left(\frac{\gamma P}{\rho}\right)^{1/2} \tag{25.2.1}$$

where γ is the ratio of the specific heats. Now for a first approximation, let's assume a constant density star. The pressure at any point in such a star can be found by directly integrating

$$\frac{dP}{dr} = -\mathcal{M}\frac{G\rho}{r^2} = -\left(\frac{4}{3}\pi r^3 \rho\right)\frac{G\rho}{r^2} = -\frac{4}{3}\pi G\rho^2 r$$

to yield

$$P(r) = \frac{2}{3}\pi G\rho^2 \left(R^2 - r^2\right)$$

where R is the stellar radius. The period of pulsation is then roughly the time it takes for this sound wave to cross the star, *i.e.*,

$$\Pi \approx 2\int_0^R \frac{dr}{v_s} \approx 2\int_0^R \left(\frac{2}{3}\gamma\pi G\rho(R^2 - r^2)\right)^{-1/2} dr \approx \left(\frac{3\pi}{2\gamma G\rho}\right)^{1/2} \tag{25.2.2}$$

There is only one variable in this equation — the density. To a good approximation, this holds true for all stars pulsating (radially) in the fundamental mode: the period of the star is inversely proportional to the square root of the average density. In other words

$$\Pi\langle\rho\rangle^{1/2} = Q \tag{25.2.3}$$

where Q is some constant.

An alternative, and perhaps clearer way of seeing this is to consider the recovery time of a pulsating star if gravity is the restoring force. Under gravity, the infall time for a pulsating star is just the freefall timescale, *i.e.*,

$$\tau_{\rm ff} \sim \left(\frac{R^3}{G\mathcal{M}} \right)^{1/2} \tag{25.2.4}$$

Since this freefall is one-half the period,

$$\Pi^2 \propto \left(\frac{R^3}{\mathcal{M}} \right) \propto \langle \rho \rangle$$

and so again $\Pi \langle \rho \rangle^{1/2} = Q$

Note that in this formulation, we can substitute for the radius in (25.2.4) using $\mathcal{L} = 4\pi R^2 \sigma T_{\rm eff}^4$, and thus obtain the period of any star in the HR diagram

$$\Pi \propto \frac{\mathcal{L}^{3/4}}{T_{\rm eff}^3 \mathcal{M}^{1/2}} \tag{25.2.5}$$

In other words, the temperature-luminosity diagram (which has mass as the third dimension) can just as easily be plotted as a period-luminosity diagram (with mass as the third dimension).

Pulsation Mechanisms

In theory, there are three mechanisms which can cause mechanical instability in a star.

The ϵ mechanism: If the center of the star is compressed slightly, the nuclear reaction rates will go up, causing an increase in expansion. The expansion can then decrease the reaction rates, cool the central core, and cause contraction.

The κ mechanism: Suppose the opacity in some region of a star were to increase with density. Upon compression, the material would absorb more energy, heat up, and expand. In the ensuing expansion, the opacity would decrease, heat would be lost from the system, and the material would fall back down. Pulsation would be driven by changes in the opacity.

The γ mechanism: If, during compression, a region of the star were to heat up less than its surroundings, heat would flow into it. This heat could then cause the region to expand, and in the expansion, the excess heat could be returned to its surroundings. The specific heat of the gas would drive pulsation.

In practice, the ϵ mechanism is *not* an effective way of driving pulsations in normal stars. Although the core is unstable to ϵ-driven pulsations, the amplitudes involved are not large enough to be detectable. The exception occurs in extremely massive ($\mathcal{M} > 90\mathcal{M}_\odot$) stars, where the sensitivity of the ϵ-mechanism to temperature is enough to cause large oscillations and possibly disrupt the star.

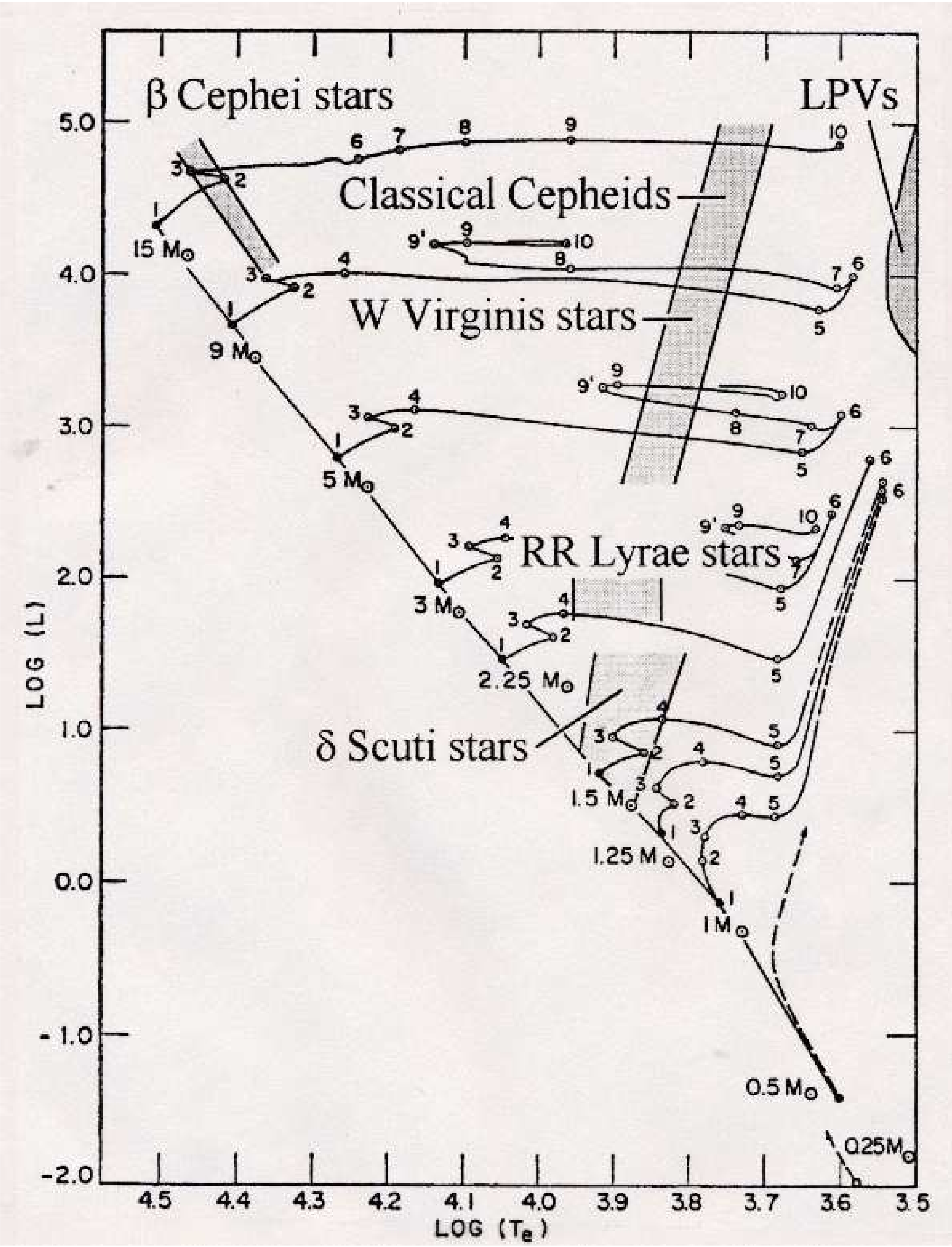
Under most circumstances, stars have Kramer-law type opacities, and have an ideal-gas equation of state. Thus

$$\kappa \propto \rho T^{-3.5} \propto \rho^{-2.5}$$

which means that the κ mechanism will not work. However, in transition regions where stellar material is only partially ionized, the energy produced by compression will go into increasing the ionization fraction, rather than the thermal motion of the particles. When this happens, the κ mechanism is effective. Moreover, during compression, this region of partial ionization will be somewhat cooler than normal, due to the energy lost to the ionization process. Thus, heat will flow into the region and the γ mechanism will also operate.

The κ and γ mechanisms can only drive pulsations in certain regions of the HR diagram. For pulsations to occur, there must be a region in the star where a substantial fraction of the hydrogen (or helium) is partially ionized. If the star is too hot, this zone will be located very near the stellar surface, where the density is too low to drive stellar oscillations. On the other hand, if the star is too cool, convection will occur at the surface. Since the energy transported by convection is proportional to the amount of matter being moved, during compression more material will move, the heat flow will increase, and the effectiveness of the energy damming will be decreased. Thus, there is an "instability strip" in the HR diagram.

(Actually, there are several instability strips in the HR diagram. The classic instability strip associated with Cepheids, RR Lyr stars, δ Scuti (main sequence) stars, and ZZ Ceti white dwarfs, is due to the partial ionization (and recombination) of He II. At the extreme red edge of the HR diagram is the hydrogen and He I instability strip; in this area are Mira stars and other Long Period Variables. Far to the blue in the HR diagram is an instability strip associated with the partial ionization of carbon and oxygen. Of course, this latter zone is not important for normal stars, since CO is usually not abundant enough to drive pulsations. However, hydrogen-deficient post-AGB stars (K1-16 type planetary nebula nuclei and PG 1159 white dwarfs) are susceptible to oscillations.)



Location of the instability strips in the HR diagram.