The Stellar Opacity

The mean absorption coefficient, κ, is not a constant; it is dependent on frequency, and is therefore frequently written as κ_ν. Inside a star, several different sources of opacity are important, and each has its own frequency dependence. To compute the appropriate "average" opacity, $\bar{\kappa}$ recall that Fick's law states that the flux at any point is related to the energy density U by

$$F_\nu = -D\,\nabla\,U = -\frac{1}{3}\bar{v}l\nabla n = -\frac{1}{3}\frac{c}{\kappa_\nu \rho}\nabla\,U$$

and that, when integrated over all energies,

$$F = -\frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{dr} \tag{3.1.4}$$

But how is $\bar{\kappa}$ related to κ_ν? To answer this question, we start by explicitly writing in the frequency dependence of U, i.e.,

$$U_\nu = \frac{4\pi}{c}B_\nu(T) = \frac{4\pi}{c}\frac{2h\nu^3}{c^2}\left\{\frac{1}{e^{h\nu/kT}-1}\right\}$$

and compute the monochromatic flux distribution

$$F_\nu = -D\,\nabla\,U_\nu = -\frac{c}{3\rho}\frac{1}{\kappa_\nu}\nabla\left\{\frac{4\pi}{c}B_\nu(T)\right\} = -\frac{4\pi}{3\rho}\frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}\frac{dT}{dr}$$

The total flux, integrated over all frequencies is then

$$F = -\frac{4\pi}{3\rho}\frac{dT}{dr}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}\,d\nu \tag{6.1.1}$$

Comparing (6.1.1) and (3.1.4), we see that

$$\frac{1}{\bar{\kappa}} = \frac{\pi}{acT^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}\,d\nu \tag{6.1.2}$$

or, since

$$\int_0^\infty \frac{dB_\nu}{dT} d\nu = \frac{d}{dT} \int_0^\infty B_\nu d\nu = \frac{dB}{dT} = \frac{ac}{\pi} T^3$$

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} \qquad (6.1.3)$$

The average opacity that you should use is therefore that which is harmonically weighted over the temperature derivative of the Planck curve. This is called the **Rosseland mean opacity.** Note that the function weights the high frequencies more than the low frequencies. To see this, we can take the temperature derivative of the Planck function

$$\frac{dB_\nu}{dT} = \frac{2h^2\nu^4}{c^2 kT^2} \frac{e^{h\nu/kT}}{\left\{ e^{h\nu/kT} - 1 \right\}^2}$$

$$= \left(\frac{2k^3 T^2}{c^2 h^2} \right) \left(\frac{x^4 e^x}{\left\{ e^x - 1 \right\}^2} \right) \qquad (6.1.4)$$

where $x = h\nu/kT$, and find its peak by taking the derivative. Then

$$\left(\frac{2k^3 T^2}{c^2 h^2} \right) \left\{ \frac{x^4 e^x + 4x^3 e^x}{(e^x - 1)^2} - \frac{2x^4 e^{2x}}{(e^x - 1)^3} \right\} = 0$$

$$x^3 e^x (e^x - 1)^{-3} (xe^x - x + 4e^x - 4 - 2xe^x) = 0$$

$$x^3 e^x (e^x - 1)^{-3} (4e^x - 4 - x - xe^x) = 0$$

Mathematically, this works out to $x \approx 3.83$. In other words, the high energy photons count much more than low energy photons.

Although in real models, the Rosseland mean opacity is computed numerically, it is instructive to evaluate the form of $\bar{\kappa}$ when $\kappa_\nu \propto \nu^{-n}$. From the definition of the Rosseland mean

$$\frac{1}{\bar{\kappa}} \propto \frac{\dfrac{2h^2}{c^2 kT^2} \displaystyle\int_0^\infty \nu^{n+4} \dfrac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}\, d\nu}{\dfrac{2h^2}{c^2 kT^2} \displaystyle\int_0^\infty \nu^4 \dfrac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}\, d\nu}$$

which, in terms of x is

$$\frac{1}{\bar{\kappa}} \propto \frac{\displaystyle\int_0^\infty \left(\frac{kT}{h}\right)^{n+4} x^{n+4} \dfrac{e^x}{(e^x-1)^2}\left(\frac{kT}{h}\right) dx}{\displaystyle\int_0^\infty \left(\frac{kT}{h}\right)^4 x^4 \dfrac{e^x}{(e^x-1)^2}\left(\frac{kT}{h}\right) dx}$$

or

$$\frac{1}{\bar{\kappa}} \propto \left(\frac{kT}{h}\right)^n \frac{\displaystyle\int_0^\infty x^{n+4} \dfrac{e^x}{(e^x-1)^2}\, dx}{\displaystyle\int_0^\infty x^4 \dfrac{e^x}{(e^x-1)^2}\, dx} \propto T^n \qquad (6.1.5)$$

Thus, if the opacity law is $\kappa \propto \nu^{-n}$, the temperature dependence of the Rosseland mean opacity will be $\bar{\kappa} \propto T^{-n}$. This follows since the mean opacity must drop as the number of high energy photons increases.

ELECTRON SCATTERING

When free electrons have thermal motions that are much less than their rest mass energy, *i.e.*, $m_e c^2 \gg kT$, or $T \ll 5.9 \times 10^9$ K, the cross-section for scattering of photons off electrons is frequency independent. The opacity per unit mass of material in this case is

$$\kappa_e = \frac{n_e \sigma_e}{\rho}$$

where σ_e is the Thomson cross-section of the electron

$$\sigma_e = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2 = 0.665 \times 10^{-24} \text{ cm}^2$$

and n_e is the density of electrons

$$n_e = \rho N_A \sum_i \left(\frac{x_i}{A_i} \right) f_i Z_i \tag{5.1.4}$$

If the gas is fully ionized, *i.e.*, if the ionization fractions $f_i \approx 1$ (as it will be in stellar interiors), and if we approximate the atomic weight (A_i) of a metal as twice its atomic number (Z_i), then

$$n_e = \rho N_A \sum_i \frac{Z_i x_i}{A_i} \approx \rho N_A \left\{ \frac{X}{1} + \frac{Y}{2} + \frac{Z}{2} \right\}$$

where X, Y, and Z are the mass fractions of hydrogen, helium, and metals, respectively. Since $X + Y + Z = 1$,

$$n_e = \rho N_A \left\{ \frac{1}{2} + \frac{1}{2} X \right\} \tag{6.1.6}$$

and

$$\kappa_e = N_A \sigma_e (1 + X)/2 = 0.2\,(1 + X) \text{ cm}^2 \text{ g}^{-1} \tag{6.1.7}$$

Note again that as long as the ionization is complete (and matter is not degenerate), the result is independent of temperature and density.

FREE-FREE ABSORPTION

The energy emitted per unit volume by a single electron scattered by ions of charge Z (as a function of velocity, v) is

$$\frac{dE}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m_e^2 v} n_e n_I Z^2 g_{\text{ff}}(v, \omega)$$

where e is the charge of the electron, and g_{ff} is the gaunt factor. To get the emissivity of thermal bremsstrahlung as a function of frequency, we integrate this over a Maxwellian distribution, noting that to get a photon with frequency ν, the incident electron must have $m_e v^2/2 \geq h\nu$. Thus, $v_{\min} = (2h\nu/m_e)^{1/2}$ and

$$\epsilon_{\text{ff}}(\nu) = 4\pi j_{\text{ff}} = \int_{v_{\min}}^{\infty} \frac{dE}{d\omega dV dt} 4\pi \left(\frac{m_e}{2\pi kT}\right)^{3/2} v^2 e^{-m_e v^2/2kT} dv$$

$$= \frac{2^5 \pi e^6}{3 m_e c^3} \left(\frac{2\pi}{3 m_e kT}\right)^{1/2} g_{\text{ff}} \, n_e \, n_I \, Z^2 \, e^{-h\nu/kT}$$

Next, to go from emission to absorption, we can use the fact that in thermodynamic equilibrium, Kirchoff's law must hold, hence the source function is $S_\nu = j_\nu/a_\nu = B_\nu(T)$, where a_ν, is the absorption coefficient per electron (with units of inverse distance). Thus

$$a_{\text{ff}}(\nu) = \rho \kappa_\nu = \frac{4 e^6}{3 m_e h c} \left(\frac{2\pi}{3 m_e kT}\right)^{\frac{1}{2}} g_{\text{ff}} \, n_e \, n_I \, Z^2 \, \nu^{-3} \left\{1 - e^{-h\nu/kT}\right\}$$

$$(6.1.8)$$

or, upon substituting $\rho = n_e \mu_e/N_A$ and $\rho = n_I \mu_I/N_A$

$$\kappa_\nu = \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3m_e k}\right)^{\frac{1}{2}} \frac{N_A^2 Z^2}{\mu_e \mu_I} g_{\text{ff}}\ \rho\, T^{-1/2} \nu^{-3}\left\{1 - e^{-h\nu/kT}\right\}$$

$$(6.1.9)$$

The first term in the equation gives the total absorption coefficient; the exponential term at the end represents a negative contribution from stimulated emission.

If we combine the constants, letting

$$C = \frac{4e^6}{3m_e hc} N_A^2 \left(\frac{2\pi}{3m_e k}\right)^{1/2} = 1.34 \times 10^{56}\quad\text{(cgs units)}$$

then

$$\kappa_\nu = C\left(\frac{Z^2}{\mu_e \mu_I}\right) g_{\text{ff}}\ \rho\, T^{-\frac{1}{2}}\nu^{-3}\left\{1 - e^{-h\nu/kT}\right\} \qquad (6.1.10)$$

To calculate the Rosseland mean free-free opacity, the inverse of this expression, weighted by the derivative of the Planck Function, must be integrated over all frequencies. This is tedious, but note that if we neglect the effects of stimulated emission, the frequency dependence of the opacity is ν^{-3}, which, through (6.1.5), implies a temperature dependence of T^{-3}. This, plus the $T^{-1/2}$ dependence which is already in the equation, gives a total temperature dependence of $T^{-7/2}$. Numerically,

$$\kappa_{\text{ff}} \sim 10^{23} \frac{Z^2}{\mu_e \mu_I} \rho\, T^{-7/2}\ \text{cm}^2\ \text{g}^{-1} \qquad (6.1.11)$$

Opacities of the form $\kappa \propto \rho T^{-7/2}$ are called **Kramers opacities**.

BOUND-FREE ABSORPTION

For a hydrogenic atom, the cross section for absorption from state (n, l) to the continuum is

$$\sigma_{\text{bf}}(Z, n, l) = \frac{2^9 \pi^7 m_e e^{10} Z^4}{3\sqrt{3} \, c \, h^6 n^5} \frac{g_{\text{bf}}}{(2\pi\nu)^3} = \frac{64\pi^4 m_e e^{10}}{3\sqrt{3} \, c \, h^6} \frac{Z^4}{n^5} \frac{g_{\text{bf}}}{\nu^3}$$

(6.1.12)

where the energy of the incident photon must be great enough to ionize the atom, $i.e.$,

$$h\nu > \chi_n = \frac{2\pi^2 m_e e^4}{h^2} \frac{Z^2}{n^2}$$

From this, it is clear that the opacity law for a single species of atom will have a number of absorption "edges" below which the absorption cross section will drop quickly. The total bound-free opacity, of course, will be the sum over all elements, Z, all ionization states, i, and and all excitation states, n. In the hydrogenic ion approximation, the opacity is

$$\kappa_\nu = \sum_Z \sum_n \sigma_{\text{bf}}(Z, n, l) \left(\frac{N_{i,n}}{N_i}\right)_Z \left(\frac{N_i}{\sum N_i}\right)_Z \frac{\rho N_A}{A_Z}$$

(6.1.13)

This summation, along with the process of taking the Rosseland mean, will act to smear out many of the edges.

The bound-free opacity must be computed numerically, but we can get a rough idea of how bound-free opacity behaves by making the assumption that in a star's interior, all ions will be very highly ionized and thus will have at most one electron. The Saha equation states that the number of atoms in ionization state i is related to the number in ionization state $i + 1$ by

$$\frac{N_i}{N_{i+1}} = \frac{N_e}{2} \frac{u_i(T)}{u_{i+1}(T)} \left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} e^{\chi_i/kT}$$

(5.2.5)

where $u(T)$ are the partition functions and χ_i is the ionization energy. If the gas is highly ionized, $N_{i+i} \approx 1$, $u_{i+1} = 1$, and $u_i = 2$, and

$$N_i \approx N_e \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2} e^{\chi_i/kT}$$

We can then estimate the number of these atoms in excitation state n from the Boltzmann equation. For hydrogenic ions, the statistical weight of a level is n^2, so if we let $\chi_{i,n} = \chi_i + \chi_n$, then the total number of absorbing ions is

$$N_{i,n} \sim N_e\, n^2 \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2} e^{\chi_{i,n}/kT}$$

The total absorption coefficient is therefore

$$a_\nu \approx \sigma_{\text{bf}}\ N_{i,n}\ \frac{\rho X_Z N_A}{A_Z}$$

$$\approx \frac{64\pi^4 m_e e^{10}}{3\sqrt{3}\,c\,h^6} \frac{Z^4}{n^5} \frac{g_{\text{bf}}}{\nu^3} \left(\frac{\rho^2 N_A^2 X_Z}{\mu_e A_Z} \right) n^2 \left(\frac{h^2}{2\pi m_e kT} \right)^{\frac{3}{2}} e^{\chi_{i,n}/kT}$$

$$\approx \frac{16\pi e^6 Z^2 g_{\text{bf}}}{3 m_e c\,h\,n\nu^3} \left(\frac{2\pi^2 m_e e^4}{h^2 kT} \frac{Z^2}{n^2} \right) \frac{\rho^2 N_A^2 X_Z}{\mu_e A_Z} \left(\frac{1}{6\pi m_e kT} \right)^{\frac{1}{2}} e^{\chi_{i,n}/kT}$$

$$\approx \frac{16\pi e^6 Z^2 g_{\text{bf}}}{3 m_e c\,h\,n\sqrt{6\pi m_e k}} \left(\frac{\chi_{i,n}}{kT} e^{\chi_{i,n}/kT} \right) \left(\frac{\rho^2 N_A^2 X_Z}{\mu_e A_Z} \right) T^{-\frac{1}{2}} \nu^{-3}$$

and the opacity is

$$\kappa_{\text{bf}} = a_\nu/\rho \approx \sum_{z,n} C \left(\frac{\chi_{i,n}}{kT} e^{\chi_{i,n}/kT} \right) \left(\frac{Z^2 X_Z}{\mu_e A_Z} \right) \rho\, T^{-\frac{1}{2}} \nu^{-3}$$

$$(6.1.14)$$

where we have dumped the many constants into the variable C (and have ignored the stimulated emission term).

Now examine the factor $\chi_{i,n}/kT$. If for a particular n and Z, $\chi_{i,n} \ll kT$, then that term will contribute little to the opacity, since the species will have too few bound electrons. Similarly, if $\chi_{i,n} \gg kT$, there will not be many photons of the requisite energy to be absorbed, and the contribution to the opacity will again be small. Thus, only those terms with $\chi_{i,n} \sim kT$ will contribute. This leaves $\kappa \propto \rho\, T^{-1/2} \nu^{-3}$, which is the same as free-free opacity. Bound-free absorption will therefore also follow the Kramers opacity law.

A rough numerical estimate for the Rosseland mean for bound-free opacity is

$$\bar{\kappa}_{\mathrm{bf}} \sim 10^{25}\, Z(1+X)\, \rho\, T^{-7/2} \ \mathrm{cm}^2 \ \mathrm{g}^{-1} \qquad (6.1.15)$$

Note that this is a factor of ~ 100 greater than the Rosseland mean for free-free opacity.

BOUND-BOUND ABSORPTION

At extremely high temperatures, $T > 10^6$ K, the radiation field will consist mostly of ionizing photons, hence bound-bound absorption will be minimal ($< 10\%$). However, at cooler temperatures the onset of bound-bound transitions in the UV and far UV can double the stellar opacity.

Obviously, the computation of bound-bound opacity must be done numerically, since it involves the summation of millions of individual absorption lines. It should be noted, however, that in the interior of a star, the pressure (and thermal) broadening terms will be substantial, so the net result will be continuum absorption, rather than a series of absorption lines. Also, the expression for the absorption coefficient of hydrogenic bound-free transitions (6.1.12) is the same as that for bound-bound transitions. Thus, we can expect the same $\sim \nu^{-3}$ dependence and therefore a Kramer-like opacity law. The strength of the total absorption, however, will be critically dependent on temperature.

H$^-$ OPACITY

A free electron passing by a neutral hydrogen atom can produce a dipole moment in the atom. At that time, an extra electron can attach itself to the system; the result is an H$^-$ ion. This ion is fragile – it has an ionization energy of only 0.754 eV, and a large cross section for absorption, either through bound-free absorption

$$H^- + h\nu \rightleftharpoons H + e^-(v)$$
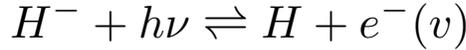
or free-free absorption

$$H^- + e^-(v) + h\nu \rightleftharpoons H^- + e^-(v)$$
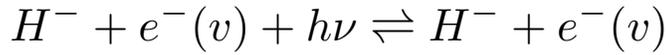
In the envelopes of cool stars, H$^-$ can be the dominant source of opacity.

The contribution of H$^-$ to the star's opacity will depend on the number of H$^-$ ions, which can be calculated from the Saha equation.

$$\frac{n_{H^-}}{n_{H_0}} = \frac{n_e}{2} \frac{u_{H^-}}{u_{H_0}} \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2} e^{\chi/kT} \qquad (5.2.5)$$

where $u_{H^-} = 1$, $u_{H_0} = 2$, and $\chi = 0.754$ eV. If x is the fraction of ionized hydrogen ($x \sim 0$ for cool stars), and a_ν is the frequency dependency of H$^-$ absorption, then the opacity due to H$^-$ is

$$\kappa_{H^-} = \frac{a_\nu n_{H^-}}{\rho} = n_H (1-x) \left(\frac{n_{H^-}}{n_{H_0}} \right) \left(\frac{a_\nu}{\rho} \right)$$

$$= a_\nu X N_A (1-x) \frac{n_e}{4} \left(\frac{h^2}{2\pi m_e kT} \right)^{\frac{3}{2}} e^{\chi/kT}$$

$$= a_\nu X N_A \frac{(1-x)}{4} \frac{P_e}{\mu_e kT} \left(\frac{h^2}{2\pi m_e kT} \right)^{\frac{3}{2}} e^{\chi/kT} \qquad (6.1.16)$$

Note that the total opacity is directly proportional to the electron density. In extremely metal poor stars, the free electrons must come from ionized hydrogen, hence the opacity will rise with temperature, until the number of neutrals starts to decline. However, for normal metal rich stars, common, low ionization metals (such as Ca, Na, K, and Al) contribute to the supply of free electrons, thereby lessening this dependence. Consequently, H^- opacity is much stronger in metal rich stars than in metal poor stars, and is stronger in dwarfs than in giants (due to the pressure term).

The determination of the frequency dependent cross-section of the H^- ion involves the quantum mechanical computation of overlapping wave functions. It is not a power law: it increases linearly at short wavelengths until ~ 8500 Å, then declines until the ionization threshold at $\sim 16,500$ Å. After that, that opacity increases again due to free-free opacity. As a result, the Rosseland mean of H^- opacity is not a Kramers law opacity. Very roughly

$$\bar{\kappa}_{H^-} \sim 2.5 \times 10^{-31} (Z/0.02) \, \rho^{1/2} \, T^9 \ \mathrm{cm^2 \ g^{-1}} \qquad (6.1.17)$$

FIGURE 4-2
Bound–free and free–free opacity from H⁻ at $T = 6300$ K.   *Ordinate:* cross-section ($\times 10^{26}$) per neutral H atom and per unit electron pressure $p_e \equiv n_e kT$;   *abscissa:* $\lambda/1000$ where λ is in Å.
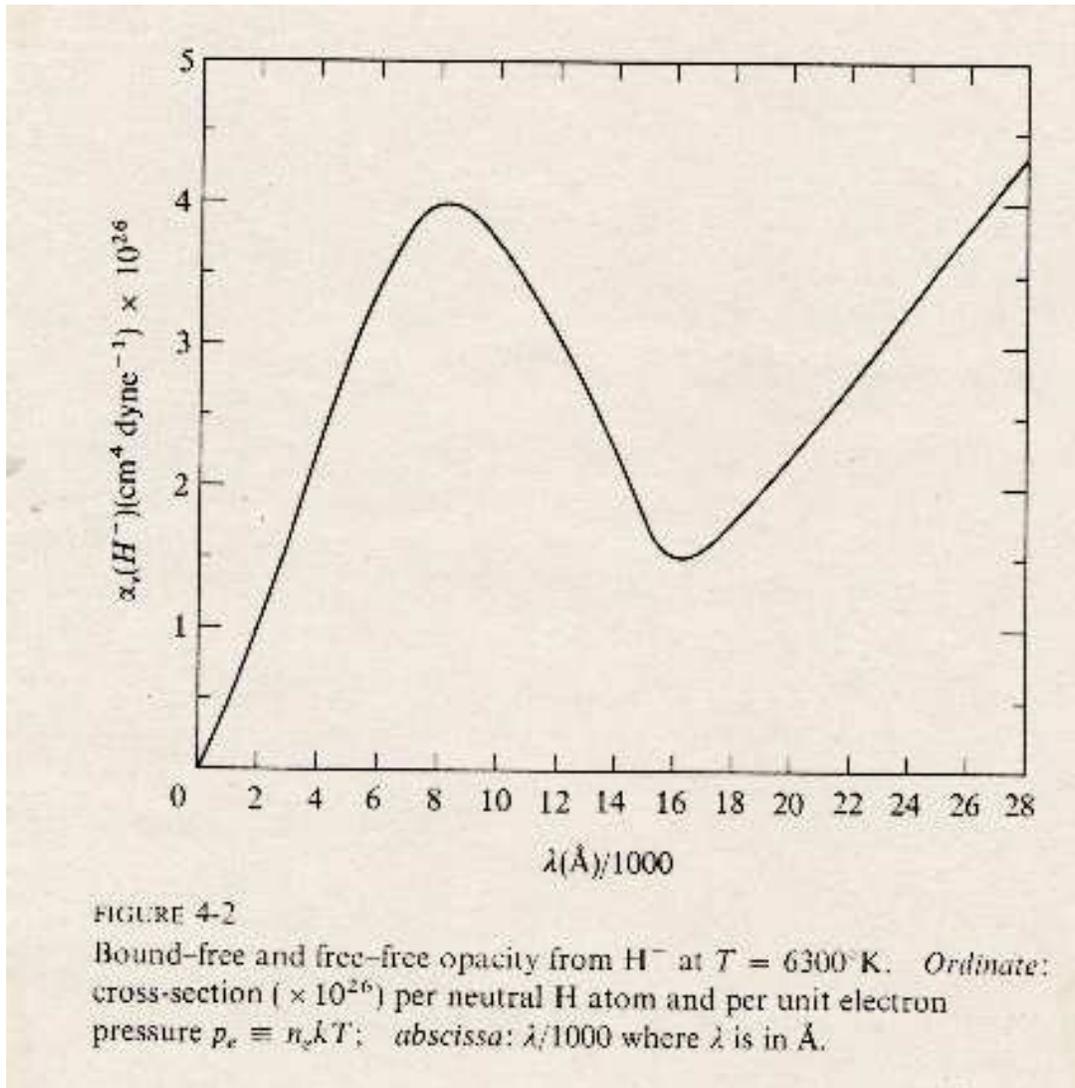
The combined bound-free and free-free opacity from the H⁻ ion at $T = 6300$ K. The y-axis plots the cross-section ($\times 10^{-26}$ per neutral H atom per unit electron pressure, $P_e = n_e kT$). From Mihalas, *Stellar Atmospheres (2nd edition)*, pg. 103.
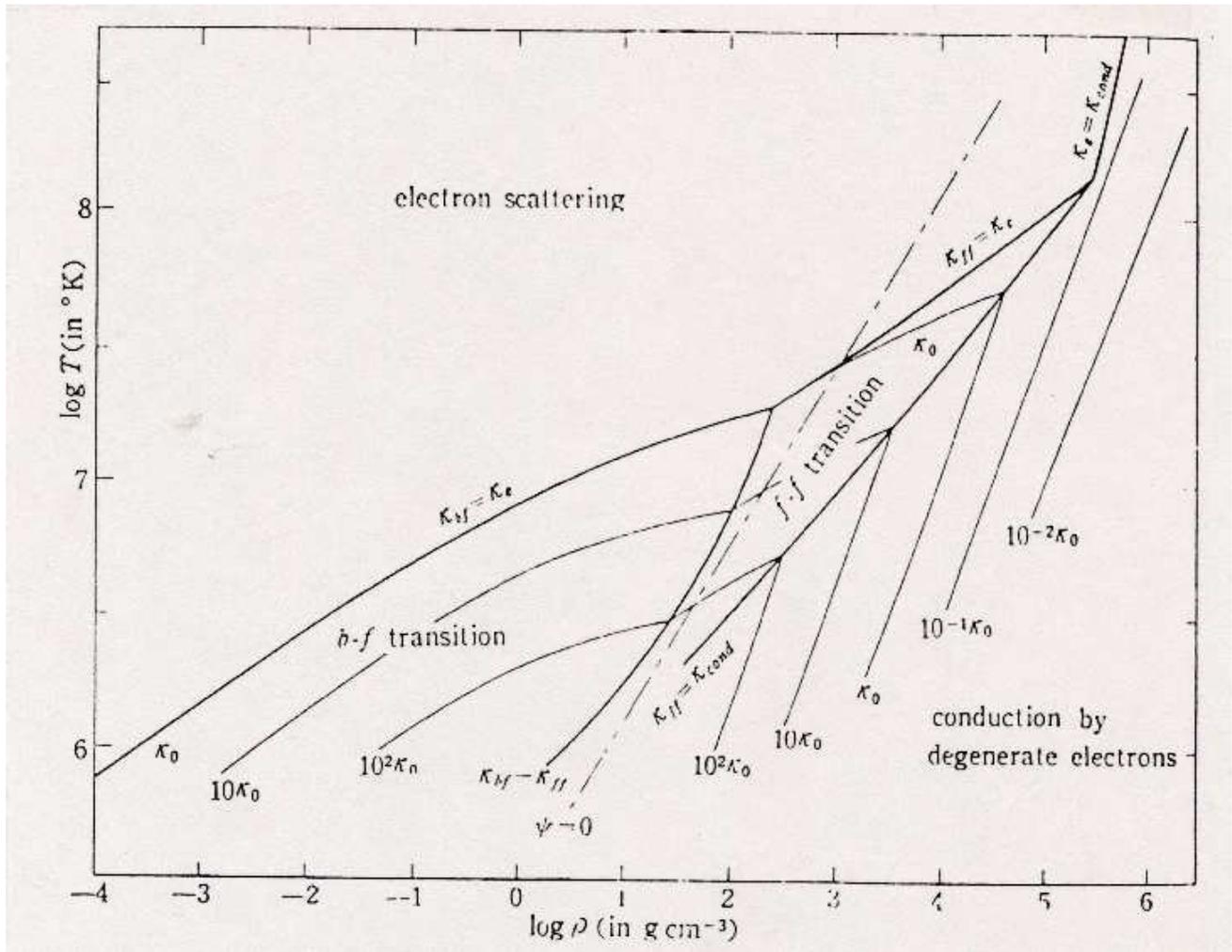
This figure illustrates the importance of the various opacities, as a function of temperature and density, for a normal (Population I) mix of abundances. The lines are labeled in units of electron scattering opacity, $\kappa_0 = \kappa_e = 0.2(1 + X)$. From Hayashi, Höshi, & Sugimoto 1962, *Suppl. Prog. Theo. Phys. Japan,* Vol. 22.
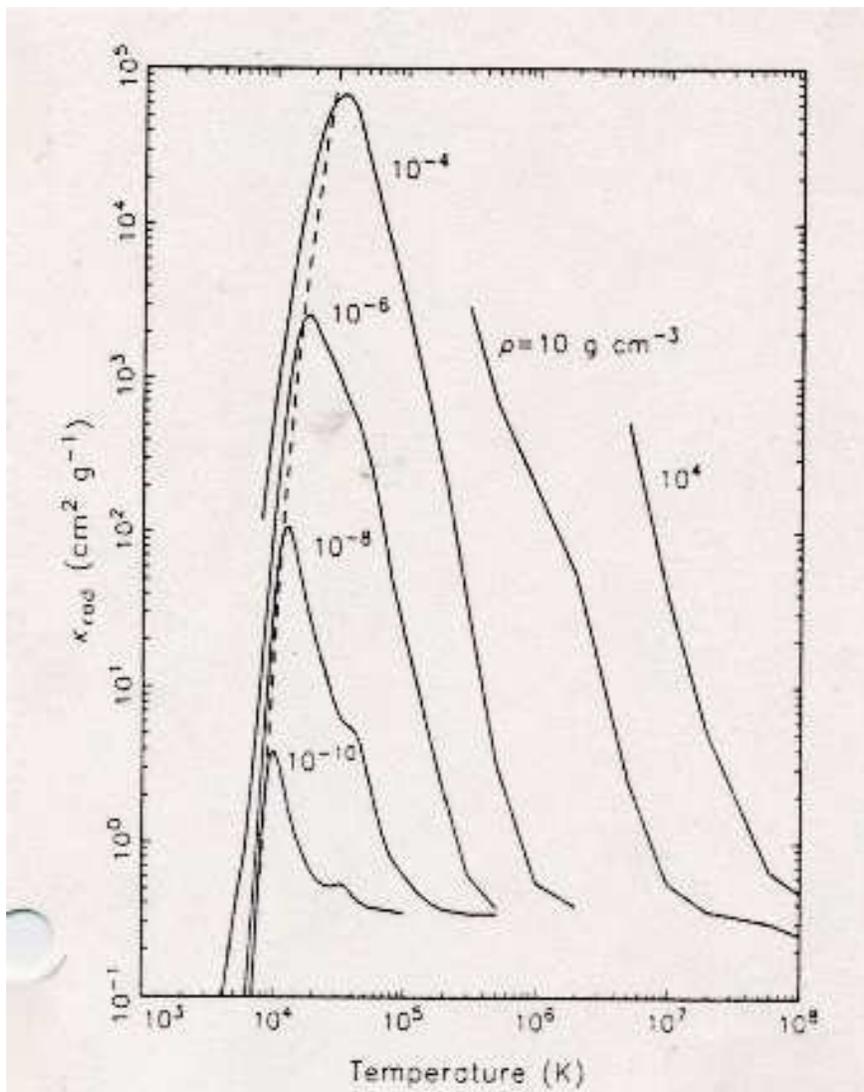
The Los Alamos National Labs radiative opacities for $X = 0.7$, $Y = 0.28$, and $Z = 0.02$. The dashed line shows the half-ionization curve for pure hydrogen.
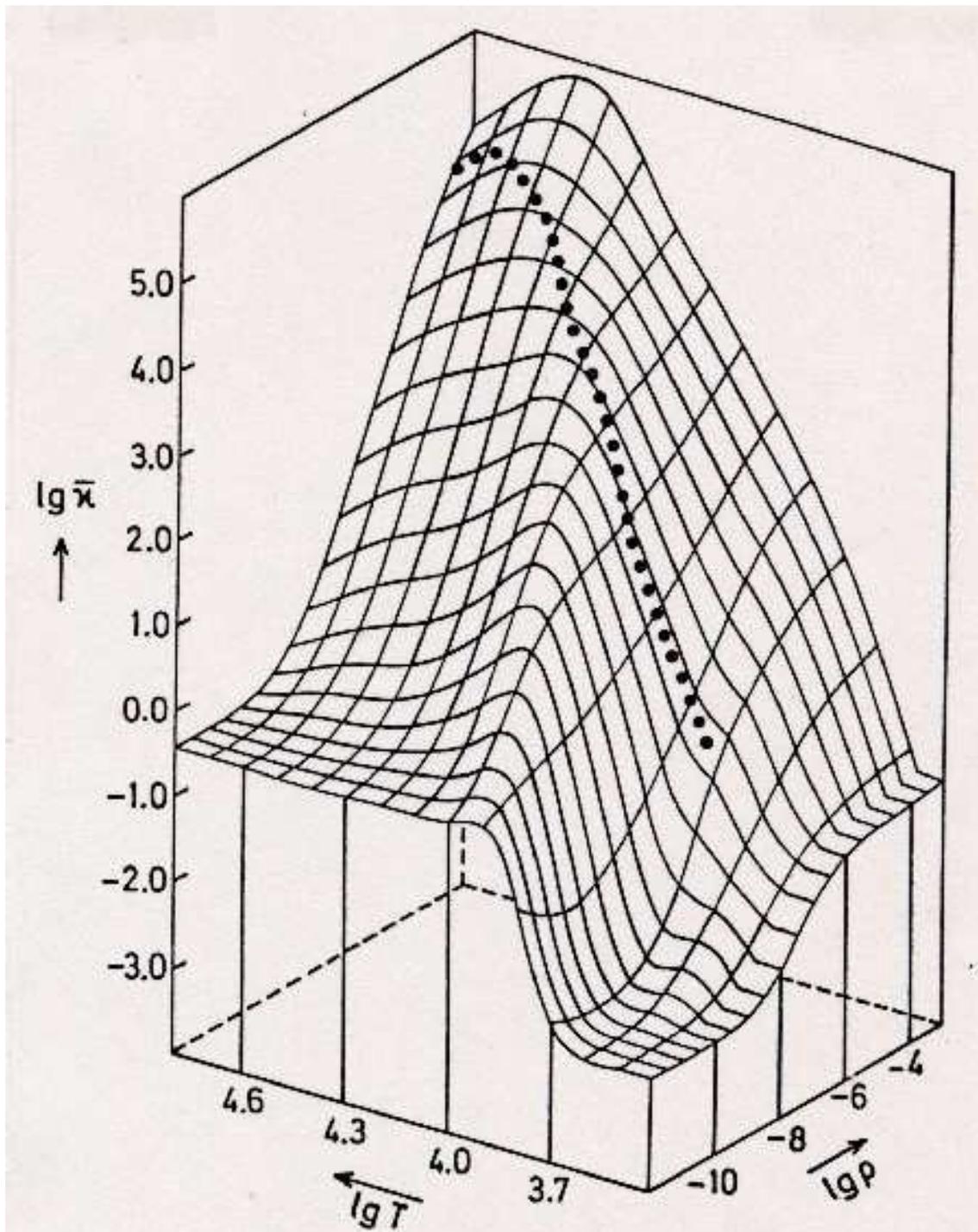
The Rosseland mean opacity (in cm^2 g^{-1}) as a function of density (in g cm^{-3}) and temperature for $X = 0.739$, $Y = 0.24$, and $Z = 0.24$. The range of values plotted is applicable for the outer regions of stars; the dots represent values in the solar model.

The Eddington Luminosity

If energy transport at the surface of a star is through radiation, then equation (3.1.4) holds, and the star's luminosity is related to its temperature gradient by

$$F = \frac{\mathcal{L}_T}{4\pi R^2} = -\frac{4ac}{3\kappa\rho} T^3 \frac{dT}{dr} \qquad (6.2.1)$$

or

$$\mathcal{L}_T = -\frac{16\pi ac}{3\kappa\rho} R^2 T^3 \frac{dT}{dr} \qquad (6.2.2)$$

Now assume that the dominate source of pressure is from radiation. (At the surface of a bright star, this will certainly be true.) For radiation pressure,

$$P_{\text{rad}} = \frac{1}{3} a T^4 \qquad (6.2.3)$$

which implies

$$\frac{dP}{dT} = \frac{4}{3} a T^3 \qquad (6.2.4)$$

and

$$\mathcal{L}_T = \left(-\frac{16\pi ac R^2 T^3}{3\kappa\rho} \right) \frac{dT}{dr} \left(\frac{3}{4aT^3} \right) \frac{dP}{dT} = -\frac{4\pi c}{\kappa\rho} R^2 \frac{dP}{dr} \quad (6.2.5)$$

Now if the star is in hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{G\mathcal{M}_T}{R^2} \rho \qquad (2.2.2)$$

which implies

$$\mathcal{L}_T = \frac{4\pi c \, G\mathcal{M}_T}{\kappa} \qquad (6.2.6)$$

This is the Eddington luminosity. If the luminosity of the star is greater than this, the surface cannot be stable, and will be ejected from the star via radiation pressure.

Note that equation (6.2.6), coupled with the mass-luminosity relation on the main sequence, implies that there is an upper limit to the mass of main sequence stars. If, for main sequence stars, $\mathcal{L}_T \propto \mathcal{M}^{3.5}$ and electron scattering is the dominant source of opacity at the surface, then

$$\left(\frac{\mathcal{M}}{\mathcal{M}_\odot} \right)^{2.5} \lesssim \frac{4\pi G c}{0.2(1+X)} \frac{\mathcal{M}_\odot}{\mathcal{L}_\odot} \tag{6.2.7}$$

This equation implies that the maximum mass of a main-sequence star is $\mathcal{M}_{max} \sim 60 \mathcal{M}_\odot$. Stars more massive than this must have strong stellar winds and will lose mass rapidly.