The Equation of State

The equation of state is the function that relates the pressure to the density, molecular weight, and temperature at any place in the star. Since it is a solely an internal property of the gas, it can, in principle, be computed once externally, and used via a lookup table, *i.e.*, $P = P(\rho, \mu, T)$.

The net pressure can be divided into three components, pressure from radiation, pressure from ions, and pressure from electrons. The latter may not obey the ideal gas law due to the effects of degeneracy.

Radiation Pressure

Normally, the pressure due to radiation in stars is small. However, in the centers of massive stars, the energy generation is large enough to make the term substantial. To compute the radiation pressure, recall that from kinetic theory, the pressure of an isotropic gas comes from the transfer of momentum, *i.e.*,

$$P = \frac{1}{3} \int n(p) \cdot v \cdot p \, d^3 p$$

Since the energy density of a photon gas is

$$u_\nu d\nu = \frac{4\pi}{c} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} d\nu \tag{7.1.1}$$

then the number density of photons, $n = u_\nu / h\nu$, in terms of the momentum, $p = h\nu/c$, is

$$n(p) \, dp = \frac{8\pi}{h^3} \frac{p^2}{e^{h\nu/kT} - 1} \, dp$$

The pressure from the photon gas is therefore

$$P_{\rm rad} = \frac{1}{3} \int n(p) \, c \, p \, dp = \frac{1}{3} \int \frac{8\pi c}{h^3} \frac{p^3}{e^{h\nu/kT} - 1} \, dp$$

or, letting $x = pc/kT$ (*i.e.*, $p = kTx/c$),

$$P_{\rm rad} = \frac{8\pi c}{3h^3} \left(\frac{kT}{c} \right)^4 \int_0^\infty \frac{x^3}{e^x - 1} \, dx \tag{7.1.2}$$

The integral in (7.1.2) can be evaluated analytically (although I wouldn't recommend it), and is

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

Thus

$$P_{\rm rad} = \left(\frac{8\pi^5 k^4}{45 c^3 h^3} \right) T^4 = \frac{aT^4}{3} \tag{7.1.3}$$

where a is the radiation constant ($a = 4\sigma/c = 7.56471 \times 10^{-15}$ ergs cm^{-3} $^\circ$K^{-4}).

Ion Pressure

Under almost all circumstances, the pressure from ions can be evaluated through the ideal gas law

$$P_{\rm ion} = \frac{\rho}{\mu_i m_a} kT \tag{7.2.1}$$

The exceptions are

Ion Crystallization. At extremely high densities and low temperatures, Coulomb interactions between ions can become important. The ions are then constrained to move within a lattice, and their properties change. This should happen when the thermal energy of each ion becomes less than the inter-ion Coulomb energy, *i.e.*,

$$\frac{3}{2} kT \approx \frac{(Ze)^2}{r_{\rm ion}}$$

If we note that the volume per ion, $V = 4/3\,\pi r_{\rm ion}^3 = 1/n_{\rm ion}$, then the ratio of Coulomb energy to thermal energy is

$$\Gamma_C = \frac{u_C}{u_t} = \frac{(Ze)^2}{kT}\,\frac{2}{3}\left(\frac{4}{3}\pi n_{\rm ion}\right)^{1/3} = 1.8 \times 10^{-3}\,\frac{Z^2 n_{\rm ion}^{1/3}}{T} \tag{7.2.2}$$

(Actually, crystallization begins to become important when $\Gamma_C \sim 100$.) The melting temperature is thus

$$T_m \sim 2.3 \times 10^3 Z^2 \mu^{-1/3} \rho^{1/3} \tag{7.2.3}$$

Except for evolved white dwarfs, stellar temperatures are always greater than the melting temperature.

Ion Degeneracy. Because the mass of an ion is $\gtrsim 2000$ times that of an electron, the momentum vectors associated with ions are much larger. (In the case of protons, by a factor of $\sqrt{m_p/m_e}$.) Thus, the phase space available for ions is much greater than for electrons, and ion degeneracy is not a problem. Note also that α-particles are not Fermions, and thus degeneracy rules do not apply to compact helium cores.

Electron Pressure

For main sequence stars, the pressure from electrons obeys the ideal gas law. But for more evolved stars, electron degeneracy may be important. To understand how and why degeneracy works, first compare the Maxwellian velocity distribution

$$f(p)\,dp\,dV = n_e\,\frac{4\pi p^2}{(2\pi m_e kT)^{3/2}}\,\exp\left(-\frac{p^2}{2m_e kT}\right)dp\,dV \qquad (7.3.1)$$

with the Pauli exclusion principle. According to Pauli, each electron occupies a region of phase space

$$dV\,dp = (\Delta x\,\Delta p_x)\,(\Delta y\,\Delta p_y)(\Delta z\,\Delta p_z) \sim h^3$$

and only two electrons (spin-up, spin-down) can occupy the same region of phase space at the same time. Therefore, since the volume of phase space available to electrons with momenta between p and $p+dp$ is $4\pi p^2 dp$, the maximum phase-space density of these electrons is

$$f(p)\,dp\,dV \leq \frac{2\,(4\pi p^2)}{h^3}dp\,dV \qquad (7.3.2)$$

For high densities, the number of electrons at a given momentum state may exceed the exclusion condition. The electrons are then forced into higher momentum states, until the Fermi momentum, p_f is reached, *i.e.*,

$$n_e\,dV = \int_0^{p_f}\frac{8\pi p^2}{h^3}\,dp\,dV = \frac{8\pi}{3h^3}\,p_f^3\,dV \qquad (7.3.3)$$

This equation defines the Fermi momentum. It is the minimum momentum required to fit $n_e\,dV$ electrons into their lowest states.

COMPLETELY DEGENERATE, NON-RELATIVISTIC GAS

The equation of state for a completely degenerate non-relativistic gas (*i.e.*, one in which $p_f \ll m_e c^2$) is fairly straightforward. When degenerate, the number of electrons with momenta between p and $p + dp$ is

$$f(p) = \begin{cases} 8\pi p^2/h^3 & \text{for } p \leq p_f \\ 0 & \text{for } p > p_f \end{cases} \tag{7.3.4}$$

To calculate the pressure (the flux of momenta through a unit surface), you need to know two things:

1) The flux of electrons hitting the surface, and
2) How much momentum is transferred by each electron.

The former quantity can be calculated in a manner similar to diffusion. Recall that, for diffusion, we said that the flux of particles with velocity v diffusing across a boundary from one direction is

$$F_1 = \frac{1}{6} n_{z-l}\, v_{z-l} \tag{3.1.1}$$

where l is the particle mean free path and v is the particle velocity. If the mean free path is small, the total number of electrons crossing the boundary from both directions is

$$F_1 + F_2 = \frac{1}{6} n_{z-l}\, v_{z-l} + \frac{1}{6} n_{z+l}\, v_{z+l} \approx \frac{1}{3} n v$$

If each electron carries an amount of momentum, p, then the pressure at the boundary from electrons with velocity v is

$$P = \frac{1}{3}\, n(v)\, v\, p$$

and the total pressure is

$$P = \frac{1}{3} \int_0^\infty n(v)\, p\, v\, dv = \frac{1}{3} \int_0^\infty n(p)\, p\, v\, dp \tag{7.3.5}$$

For a non-relativistic electron gas, $v = p/m_e$, so for complete degeneracy

$$P_e = \frac{1}{3}\int_0^{p_f} \frac{8\pi p^2}{h^3}\, p\left(\frac{p}{m_e}\right) dp = \frac{8\pi}{3m_e h^3}\int_0^{p_f} p^4 dp = \frac{8\pi}{15 m_e h^3}\, p_f^5$$

If we then substitute for p_f using (7.3.3)

$$p_f = \left(\frac{3h^3}{8\pi}\cdot\frac{\rho}{\mu_e m_a}\right)^{1/3} \tag{7.3.6}$$

and

$$P_e = \frac{h^2}{20 m_e m_a}\left(\frac{3}{\pi m_a}\right)^{2/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

$$= 1.0036\times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad \text{cgs} \tag{7.3.7}$$

Note that there is no temperature dependence in this equation. The distribution of electrons in phase space is entirely determined by the Fermi exclusion principle, leaving pressure proportional only to density.

Since there must be a smooth transition between the degenerate and non-degenerate states, we can estimate the density where this occurs by simply equating the two expressions from pressure

$$P_{trans} = \frac{\rho}{\mu m_a} kT = \frac{h^2}{20 m_e m_a}\left(\frac{3}{\pi m_a}\right)^{2/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Degeneracy therefore becomes important where

$$\rho = \left(\frac{20 m_e k}{\mu}\right)^{3/2}\left(\frac{\pi m_a}{3h^3}\right)\mu_e^{5/2}\, T^{3/2}$$

$$= 23.9\left(\frac{T}{10^6}\right)^{3/2} \mu_e^{5/2}\,\mu^{-3/2} \text{ g cm}^{-3} \tag{7.3.8}$$

COMPLETELY DEGENERATE, RELATIVISTIC GAS

If the density is extremely high, the velocities that the electrons must have to obey the Pauli exclusion rule will be relativistic. In this case, we must substitute for velocity using

$$p = \frac{m_e v}{\sqrt{1 - v^2/c^2}} \tag{7.3.9}$$

The integral for pressure is then slightly more complicated

$$P_e = \frac{1}{3} \int_0^{p_f} \frac{8\pi p^2}{h^3} p \left(\frac{p/m_e}{\sqrt{1 + p^2/(m_e^2 c^2)}} \right) dp \tag{7.3.10}$$

If we let $\xi = p/(m_e c)$, and $x = p_f/(m_e c)$, then the integral can be re-written as

$$P_e = \frac{8\pi m_e^4 c^5}{3h^3} \int_0^x \frac{\xi^4}{(1+\xi^2)^{1/2}} d\xi$$

This integral is analytic:

$$\begin{aligned} \int_0^x \frac{\xi^4 d\xi}{(1+\xi^2)^{1/2}} &= \frac{1}{8} x \left(2x^2 - 3\right) \left(x^2 + 1\right)^{1/2} + 3 \sinh^{-1} x \\ &= \frac{1}{8} x \left(2x^2 - 3\right) \left(x^2 + 1\right)^{1/2} + 3 \ln\{x + (x^2+1)^{\frac{1}{2}}\} \end{aligned}$$

Following Chandrasekhar's *Principles of Stellar Structure*, if we let

$$f(x) = x \left(2x^2 - 3\right) \left(x^2 + 1\right)^{1/2} + 3 \sinh^{-1} x \tag{7.3.11}$$

then

$$P_e = \frac{\pi m_e^4 c^5}{3h^3} f(x) \tag{7.3.12}$$

Note that if the gas is extremely relativisitic, $p_f \gg m_e c$, thus $x \gg 1$, and $f(x) \to 2x^4$. In this limiting case

$$P_e = \frac{\pi m_e^4 c^5}{3h^3} 2 \left(\frac{p_f}{m_e c} \right)^4 = \frac{2\pi c}{3h^3} p_f^4$$

Substituting density for p_f using (7.3.6), we get

$$\begin{aligned} P_e &= \left(\frac{hc}{8m_a} \right) \left(\frac{3}{\pi m_a} \right)^{1/3} \left(\frac{\rho}{\mu_e} \right)^{4/3} \\ &= 1.2435 \times 10^{15} \left(\frac{\rho}{\mu_e} \right)^{4/3} \quad \text{cgs} \end{aligned} \tag{7.3.13}$$

Again, the pressure is independent of temperture, but this time, the exponent is 4/3.

The transition point between the relativisitic and non-relativistic cases occurs when $p_f \sim m_e c$. If we substitute $m_e c$ into (7.3.6), we find that this occurs at

$$\rho > \frac{8\pi \mu_e m_a}{3h^3} m_e^3 c^3 = 9.7 \times 10^5 \text{ g cm}^{-3} \tag{7.3.14}$$

PARTIAL DEGENERACY

In many cases, when both the temperature and density are high, partial degeneracy occurs. When this happens, the electron distribution in phase space is neither Maxwellian

$$f^M(p)\,dp\,dV = n_e \frac{4\pi p^2}{(2\pi m_e kT)^{3/2}} \exp\left(-\frac{p^2}{2m_e kT}\right) dp\,dV \quad (7.3.1)$$

nor degenerate

$$f^D(p)dp\,dV = \frac{8\pi p^2}{h^3} dp\,dV \quad p < p_f \quad (7.3.3)$$

but instead is given by

$$f^P(p)\,dp\,dV = \frac{8\pi p^2}{h^3} \frac{1}{1 + e^{E/kT-\psi}} dp\,dV \quad (7.3.15)$$

where ψ is the degeneracy parameter,

$$\psi = \ln\left\{\frac{n_e h^3}{2(2\pi m_e kT)^{3/2}}\right\} \quad (7.3.16)$$

and E is the kinetic energy of the electron

$$E = E_{\rm tot} - m_e c^2 = m_e c^2 \left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2} - m_e c^2 \quad (7.3.17)$$

Note that this equation does have the correct form. If n_e is small, then $\psi \ll 0$, and $f^P(p) \to f^M(p)$. On the other hand, if n_e is large, $e^{E/kT-\psi} \ll 0$, and $f^P(p) \to f^D(p)$.

Note also that ψ depends only on $n_e/T^{3/2}$.

With this general expression for the electron phase-space occupation, the expressions for n_e, P_e, and u_e are

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2}{1+e^{E/kT-\psi}}dp \tag{7.3.18}$$

$$P_e = \frac{8\pi}{3h^3}\int_0^\infty \frac{p^3 v(p)}{1+e^{E/kT-\psi}}dp \tag{7.3.19}$$

$$u_e = \frac{8\pi}{h^3}\int_0^\infty E\,\frac{p^2}{1+e^{E/kT-\psi}}dp \tag{7.3.20}$$

where in the non-relativisitic case,

$$E = m_e c^2 \left\{\left(1+\frac{p^2}{m_e^2 c^2}\right)^{1/2} - 1\right\} \approx \frac{p^2}{2m_e}$$

$$v = \frac{p}{\left(m_e^2 + p^2/c^2\right)^{1/2}} \approx p/m_e \tag{7.3.21}$$

For the non-relativistic case, if we let $\eta = p^2/2m_e kT$, then these integrals become

$$n_e = \frac{4\pi}{h^3}(2m_e kT)^{3/2}\int_0^\infty \frac{\eta^{1/2}}{1+e^{\eta-\psi}}\,d\eta \tag{7.3.22}$$

$$P_e = \frac{4\pi}{3h^3}\frac{(2m_e kT)^{5/2}}{m_e}\int_0^\infty \frac{\eta^{3/2}}{1+e^{\eta-\psi}}\,d\eta \tag{7.3.23}$$

$$u_e = \frac{2\pi}{h^3}\frac{(2m_e kT)^{5/2}}{m_e}\int_0^\infty \frac{\eta^{3/2}}{1+e^{\eta-\psi}}\,d\eta \tag{7.3.24}$$

All the above integrals have the form

$$F_\nu(\psi) = \int_0^\infty \frac{\eta^\nu}{1+e^{\eta-\psi}}\,d\eta \tag{7.3.25}$$

These are called Fermi-Dirac integrals, and they are tabulated in McDougall & Stoner 1939, *Phil. Trans. R. Soc. London,* **237,** 67 (so you don't need to numerically integrate them each time).

The General Equation of State

We get the general equation of state for stellar matter by summing the expressions for radiation pressure, ion pressure, and electron pressure. Thus

$$P = \frac{aT^4}{3} + \frac{\rho}{\mu_i m_a} kT + \frac{8\pi}{3h^3} \int_0^\infty \frac{p^3\, v(p)}{1 + e^{E/kT - \psi}}\, dp \qquad (7.4.1)$$

$$\rho = \frac{4\pi}{3} (2m_e)^{3/2} \mu_e m_a \int_0^\infty \frac{E^{1/2}}{1 + e^{E/kT - \psi}}\, dE \qquad (7.4.2)$$

Thus, we have implicit formulae for the equation of state. Given ρ and T, you can solve for ψ via (7.3.16), and then P. Similarly, the equation for each particle's internal energy is

$$u = \frac{aT^4}{\rho} + \frac{3}{2} \frac{k}{\mu_i m_a} T + \frac{8\pi}{h^3 \rho} \int_0^\infty \frac{p^2 E(p)}{1 + e^{E/kT - \psi}} dp \qquad (7.4.3)$$

Unfortunately, there is no way to use these equations to generate analytical expressions for the thermodynamic quantities c_P, c_V, α, δ, and ∇_{ad}. Unless one of the terms dominates, these quantities must be computed numerically.