UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **PHAS4426**

MODULE NAME : **Advanced Quantum Theory**

DATE : **26-Apr-07**

TIME : **10:00**

TIME ALLOWED : **2 Hours 30 Minutes**

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Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

1. A system subjected to a time-dependent perturbation is described by a Hamiltonian:

$$H(\mathbf{r},t) = H_0(\mathbf{r}) + \lambda V'(\mathbf{r},t)$$

where λ is a small parameter. The eigenfunctions $\psi_n^{(0)}(\mathbf{r})$ and eigenvalues E_n of $H_0(\mathbf{r})$ are known. Given that a solution of the time-dependent Schrödinger equation can be written as:

$$\Psi(\mathbf{r},t) = \sum_n c_n(t)\psi_n^{(0)}(\mathbf{r})\exp(-iE_nt/\hbar)$$

obtain a differential equation for the transition coefficients $c_n(t)$. [4]

Initially, at time t_0, the system is in a definite eigenstate $\psi_i^{(0)}(\mathbf{r})$. Show that at a later time t, to first order in λ, the transition amplitude for excitation of a state $\psi_k(\mathbf{r})$ of energy E_k ($\neq E_i$) is given by

$$c_k(t) = \frac{1}{i\hbar}\int_{t_0}^{t} < \psi_k^{(0)}(\mathbf{r})|\lambda V'(\mathbf{r},t')|\psi_i^{(0)}(\mathbf{r}) > e^{i\omega_{ki}t'}dt'$$

where $\omega_{ki} = (E_k - E_i)/\hbar$. [5]

Show that the case $i = k$ corresponds to a simple phase shift on the initial eigenfunction. [2]

A quantum particle has eigenfunctions $\psi_m(x) = \frac{1}{\sqrt{2\pi}}e^{imx}$ where $m = 0, \pm1, \pm2...$, where the coordinate $0 \leq x \leq 2\pi$ and the energies $E_m = m\epsilon$. At $t \leq 0$ the particle is in the eigenstate corresponding to $m = +2$. At $t > 0$ it is acted on by a weak perturbation:

$$\lambda V'(\mathbf{r},t) = A\ \sin x \exp(-\gamma t)$$

Into which states can transitions be induced, to first order in A? [4]

Calculate the form of the transition probability at time $t = T$ for the allowed transitions. Discuss the behaviour as $T \to \infty$. [5]

2. Discuss briefly the regime of validity of the WKB method and its applications in quantum theory, including the significance of connection formulae. [6]

The general form of the WKB wavefunction is given as a sum of terms of the form $\frac{A_\pm}{\sqrt{K(x)}} e^{\pm i \int K(x')dx'}$. Give the corresponding form for a classically forbidden region, where $E < V(x)$, defining all terms carefully. [2]

A quantum particle is in a well with a single rigid wall at $x = 0$.

$$x < 0 \quad V(x) = \infty$$
$$L \geq x \geq 0 \quad V(x) = V_0\, x/L$$
$$x > L \quad V(x) = V_0$$

where $V_0 > 0$ is a constant.

Show that this type of well obeys the general quantization condition

$$\int_{T_1}^{T_2} K(x')dx' = (n + 3/4)\pi,$$

explaining (and giving precise values) for the limits of integration T_1 and T_2. [7]

Show that if $\sqrt{2mV_0} = 1$ and $\frac{2L}{3\hbar\pi} = 216$, the eigenenergies E_n of the particle, within the WKB approximation, are given by the relation:

$$\frac{E_n}{V_0} = \frac{1}{36}\left[n + \frac{3}{4}\right]^{2/3}.$$

[3]

How many bound states can the well support? [2]

NB: You may use $\int (1-y)^{1/2} dy = -2/3(1-y)^{3/2}$.

Recall also the connection formula, of general form:
$\frac{A}{\sqrt{q(x)}} \exp - \left[\int_x^b q(x')dx'\right] \to \frac{2A}{\sqrt{K(x)}} \cos\left[\int_b^x K(x')dx' - \pi/4\right]$.

3. We can define a quantum operator $A^\dagger$ to be the adjoint operator of another operator A if:

$$\langle A\phi|\psi\rangle = \langle\phi|A^\dagger|\psi\rangle$$

where ϕ and ψ are arbitrary quantum states. The time-evolution operator propagates a quantum state from time t_0 to a later time t:

$$\Psi(x,t) = T(t,t_0)\Psi(x,t_0),$$

Starting from an assumption of conservation of probability, show that the operator T is unitary. [4]

Hence show that $T^\dagger(t,t_0) = T(t_0,t)$ [2]

A quantum particle evolves in a time-periodic potential $V(x,t) = V(x,t+\tau)$, with period τ, too strong to be treated perturbatively. Its Floquet states take the form:

$$\Psi_n(x,t) = \exp\left(-i\epsilon_n t\right)\, U_n(x,t)$$

where $U_n(x,t) = U_n(x,t+\tau)$ and ϵ_n is a quasi-energy.

Explain briefly how Floquet states may be used to evolve a general quantum state in a time-periodic potential. You should explain how they may be obtained from the time evolution operator and explain the role of the Floquet operator $F = H - i\hbar\frac{\partial}{\partial t}$. [9]

The Hamiltonian of a quantum particle $H = H_0 + V(x,t)$, is the sum of a time-independent part, H_0, and a time-periodic potential:

$$V(x,t) = V(x,t+\tau) = Bx\cos\omega t,$$

corresponding to period $\tau = 2\pi/\omega$.

The time-independent Hamiltonian H_0 has eigenfunctions $\psi_j(x) = \frac{1}{\sqrt{2\pi}}\sin jx$ where $j = 0,1,2...$, the coordinate $0 \leq x \leq 2\pi$ and $H_0\psi_j(x) = j^2\epsilon\psi_j(x)$.

We can expand our Floquet states in a complete basis of orthonormal states by writing:

$$U_n(x,t) = \sum_{j,m} C^n_{j,m}\psi_j(x)\exp im\omega t$$

Using this basis, calculate the form of the matrix elements, $< jm|F|j'm' >$, of the operator F. [5]

Assume $< \psi_j|x|\psi_{j'} >= A\delta_{jj'}$ where A is a constant.

Recall that $\int_0^{2\pi} e^{imx}dx = 2\pi\delta_{m0}$.

4. The spins $\mathbf{S_1}$ and $\mathbf{S_2}$ of two interacting spin-1/2 particles couple to give a total angular momentum $\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$. There are simultaneous eigenstates of S_i^2 and S_{iz} where $i = 1, 2$: $|s_i m_i\rangle = |\frac{1}{2}\ \frac{1}{2}\rangle = |\alpha_i\rangle$ and $|s_i m_i\rangle = |\frac{1}{2}\ -\frac{1}{2}\rangle = |\beta_i\rangle$ such that:

$$S_i^2|s_1 m_1\rangle = s_1(s_1+1)\hbar^2|s_1 m_1\rangle$$

and,

$$S_{z1}|s_1 m_1\rangle = m_1\hbar|s_1 m_1\rangle$$

Show that the total $\mathbf{S^2}$ can be expressed, using quantum raising and lowering operators, as follows:

$$\mathbf{S^2} = \mathbf{S_1^2} + \mathbf{S_2^2} + [S_{1+}S_{2-} + S_{1-}S_{2+}] + 2S_{1z}S_{2z}$$

[4]

By considering the action of the above operator on the spin state:

$$|\chi\rangle = a|\alpha_1\rangle|\beta_2\rangle + b|\beta_1\rangle|\alpha_2\rangle,$$

show that for $|\chi\rangle = |SM_s\rangle = |10\rangle$, where $|SM_s\rangle$ are eigenstates of $\mathbf{S^2}$ and $S_z = S_{z1} + S_{z1}$, we require $a = b = 1/\sqrt{2}$. [9]

The spin operator $S_{A_1} = \frac{\hbar}{2}[2\sigma_{x1} + i\sigma_{x1}\sigma_{y1}]$ acts only on particle 1.

We wish to represent operator S_{A_1} as a 2×2 matrix, A_1, which is a sum over Pauli matrices, ie $A_1 = \sum_j \lambda_j \sigma_j$. Show that $\lambda_i = \frac{1}{2} Tr A_1 \sigma_i$ [2]

Calculate all the coefficients λ_i. [2]

Hence, or otherwise, evaluate the expectation value $\langle\chi|S_{A_1}|\chi\rangle$ [3]

Note: Spin raising and lowering operators are given by :

$S_\pm = S_x \pm iS_y$

where $S_\pm|sm\rangle = \sqrt{[s(s+1) - m(m\pm 1)]}\hbar|sm\pm 1\rangle$.

The Pauli x and y matrices can be represented by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

5. (a) In a quantum scattering experiment, a flux J_{inc} of helium atoms is incident along the $\hat{z}$ direction on to target atoms of neutral sodium. A detector, which subtends a solid angle $d\Omega$ relative to the target, is found to collect $N_D = N(\theta,\phi)d\Omega$ atoms per second.

Discuss in detail how a comparison between theoretically calculated results obtained from the Schrödinger equation and the experimental results might be undertaken. You should explain the meaning of terms such as differential cross-section $\frac{d\sigma}{d\Omega}$, total cross-section, asymptotic forms of the quantum wavefunctions and quantum scattering amplitudes $f(\theta,\phi)$ giving any relations between them. You may illustrate your explanation with diagrams. [10]

For elastic scattering, the partial wave expansion for $f(\theta)$ is

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l(k)P_l(\cos\theta),$$

where P_l is a Legendre polynomial and δ_l is a partial-wave phase-shift. The total cross-section, σ, is given by:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k).$$

(b)Explain the physical significance of the centrifugal barrier; discuss how the the elastic scattering phaseshifts $\delta_l(k)$ might be expected to vary with energy and l as a result of a centrifugal barrier. [4]

Hence derive the approximate form of the cross section at low energy $\sigma = 4\pi A^2$ where A is a scattering length. [3]

(c) For a beam of particles scattered by a repulsive potential α/r^2, it can be shown that

$$\delta_l(k) = -\frac{\pi}{4}(2l+1)\left(\left[(1+\frac{8m\alpha}{(2l+1)\hbar^2}\right]^{1/2} - 1\right).$$

Hence show that, if $8m\alpha/\hbar^2 << 1$, and noting that as $x \to 0$, $(1+x)^{1/2} \to 1 + x/2$ then the differential cross section

$$\frac{d\sigma}{d\Omega} \simeq \frac{\pi^2 m^2\alpha^2}{4\hbar^4 k^2}\frac{1}{\sin^2(\theta/2)}.$$

[3]

NOTE : **In answering this question, you may use these results :**
$\delta_l(k) \sim k^{2l+1}$.
Also $\sum_{l=0}^{\infty} P_l(\cos\theta) = \frac{1}{2\sin(\theta/2)}$.

END OF PAPER